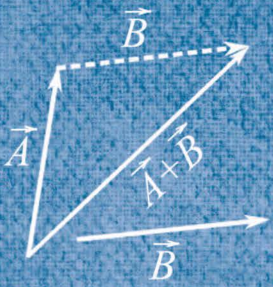

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

G.V.Rudnyeva

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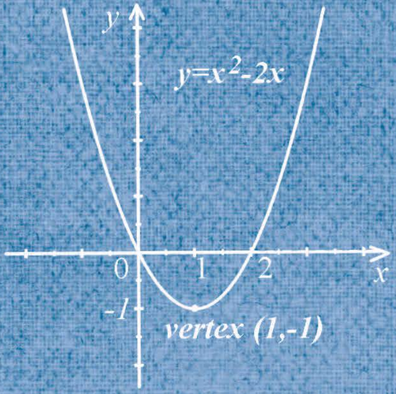

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ

НАЦІОНАЛЬНИЙ ТЕХНІЧНИЙ УНІВЕРСИТЕТ
«ХАРКІВСЬКИЙ ПОЛІТЕХНІЧНИЙ ІНСТИТУТ»

G. V. Rudnyeva

ELEMENTS OF LINEAR ALGEBRA AND ANALYTIC GEOMETRY

The Educational Textbook for the students of all technical specialties for full-time and distance education

Second Revised and Expanded Edition

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Навчальний посібник містить викладений англійською мовою теоретичний та практичний матеріал з лінійної алгебри та аналітичної геометрії. У посібнику доведені основні теореми і твердження та отримані формули, які необхідні для розв'язання практичних задач, а також подані підбірки завдань для практичних занять з кожної теми та варіанти індивідуальних завдань.

Призначено для студентів технічних університетів, що вивчають курс вищої математики англійською мовою, іноземних студентів та викладачів вищої математики.

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The book contains theoretical and practical material in linear algebra and analytic geometry in English. Theoretical part presents the proofs of the basic theorems and statements and the derivations of the formulas necessary to solve practical problems. Practical part of the book consists of practical tasks for each topic and variants of individual tasks.

It is intended for students of technical universities studying higher mathematics in English, foreign students and teachers of higher mathematics.

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INTRODUCTION

Mathematical methods are used in every field of science and engineering. This means that students, whatever their specialty, must have a solid theoretical grounding in mathematics and in solving practical problems. Moreover, since wide collaboration of Ukrainian technical universities with technical universities of other countries, it is important to study and to know the international mathematical terminology for successful work of modern engineers.

The textbook "Elements of Linear Algebra and Analytic Geometry" presents a course of mathematics suitable for under-graduate students of the technical universities and has grown out of a course of lectures given by the author at National Technical University "Kharkov Polytechnic Institute" during the past four years to the students studying the mathematical courses in English.

This textbook consists of two chapters: Chapter 1 "The Elements of Linear Algebra" and Chapter 2 "Analytic Geometry". Chapter 1 provides all necessary theory concerning to the matrices and determinants to be successful in solving and investigating the systems of linear algebraic equation. Chapter 2 deals with vector algebra, analytic geometry in space and in plane including the study of the second order curves.

The textbook contains a large number of examples and illustrations to make the subject matter readily comprehensible. Theoretical part of each chapter is followed by practical tasks for each topic and by the set of the individual tasks.

It is recommended to the students of all technical specialties who study the course of Higher Mathematics in English, to foreign students, to the lecturers of High School to help in forming their own lecture courses and practices, and also to everybody who interests in studying the mathematical terminology in English.

Author thanks very much to the Head of the Department of Applied Mathematics, Doctor of Technical Sciences, prof. Kurpa L.V. for useful discussions and advices at making up the presented textbook.

CHAPTER 1. THE ELEMENTS OF LINEAR ALGEBRA

1.1. Concept of Matrix

Investigation of many important problems in engineering leads to the solving of the systems of linear algebraic equations.

Even at school to solve some problems we had to construct and solve systems of two or three linear equations.

In general case we have to work with systems of different number of equations and unknown variables.

Moreover, the number of unknown variables does not sometimes coincide with the number of equations. And naturally some set of questions appears:

- Does the system have a solution?
- If it does then how many solutions does it have?
- How to find all possible solutions?

The goal of this chapter is to answer all these questions for any linear algebraic system of any number of unknowns.

In general case the system of m linear algebraic equations with n unknown variables has the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2, \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m, \end{cases}$$

where x with subscripts 1,2,…,n are unknown variables, a_{ij} $(i=\overline{1,m},\ j=\overline{1,n})$ are real coefficients, $b_i\,(i=\overline{1,m})$ are real numbers called free members.

The coefficients of $x_i\,(i=\overline{1,n})$ form the rectangular table. Investigations of these tables corresponding to the systems of linear equations helps to answer all formulated above questions.

Definition. The matrix of the size m by n is a rectangular table of numbers with m rows and n columns. We use the following notations to sign the matrices:

$$A = \begin{pmatrix} a_{11}\ a_{12}\ a_{13}\ \ldots\ a_{1n} \\ \cdots \\ a_{m1}\ a_{m2}\ a_{m3}\ \ldots\ a_{mn} \end{pmatrix} \text{ or } A = \left(a_{ij}\right)_{m,n}.$$

The numbers a_{ij} are called the elements of matrix. The first subscript is the number of the row and the second one is the number of the column where the element a_{ij} is situated.

m is a number of rows, n is a number of columns.

Definition. If m=n then the matrix A is called the square matrix.

Definition. If m=1 then the matrix is called the row matrix.

Definition. If n=1 then the matrix is called the column matrix.

Definition. Two matrices A and B of identical sizes are called equal if their corresponding elements are equal, i.e.

$$A=B \Leftrightarrow a_{ij} = b_{ij} \quad i = \overline{1,m},\ j = \overline{1,n}.$$

Definition. The matrix with all its elements equal to zero is called a null matrix or zero matrix.

Let us consider square matrices.

Definition. The set of the elements $a_{11}, a_{22}, \ldots, a_{nn}$ of a square matrix A is called the main (or leading) diagonal of the matrix, while these elements are called the elements of the main diagonal.

Example. Let us consider the matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 4 & 0 & 5 \\ 7 & 5 & 2 \end{pmatrix}$. The elements 1,0,2 are elements of the main diagonal.

Definition. The set of the elements $a_{1n}, a_{2n-1}, \ldots, a_{n1}$ of a square matrix A is called the secondary diagonal of the matrix, while these elements are called the elements of the secondary diagonal.

The matrix elements $-1,0,7$ from the previous example are the elements of the secondary diagonal.

Definition. A square matrix A with zero elements everywhere except in the leading diagonal is called a diagonal matrix.

Definition. The unit matrix is a diagonal matrix with all its diagonal elements equal to one.

Usual notations for such matrices are by the letters I or E.

Definition. If all the elements of a square matrix A located below (above) main diagonal are equal to zero then this matrix is called the upper (lower) triangular matrix.

Example. Suppose we have the following matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 5 & 3 & 8 \end{pmatrix},\ C = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here matrix A is upper triangular, B is lower triangular, C is diagonal and D is nor diagonal, nor unit since D is not square.

1.2. Operations on Matrices

Since the matrices are tables of numbers It is natural to introduce some algebraic operations on them such as, for example, the addition, subtraction, multiplication and division.

Definition. The algebraic sum of matrices A and B of identical sizes is called the matrix C=A+B of the same size with the elements defined as

$$c_{ij} = a_{ij} + b_{ij},\ i = \overline{1,m},\ j = \overline{1,n}.$$

Example. Let $A=\begin{pmatrix}1 & 0 & 2\\ 3 & 4 & 5\end{pmatrix}$, $B=\begin{pmatrix}3 & 1 & -2\\ -1 & 0 & 2\end{pmatrix}$. Then

$$A+B=\begin{pmatrix}1 & 0 & 2\\ 3 & 4 & 5\end{pmatrix}+\begin{pmatrix}3 & 1 & -2\\ -1 & 0 & 2\end{pmatrix}=\begin{pmatrix}1+3 & 0+1 & 2-2\\ 3-1 & 4+0 & 5+2\end{pmatrix}=\begin{pmatrix}4 & 1 & 0\\ 2 & 4 & 7\end{pmatrix}.$$

Definition. The result of multiplying a matrix $A=(a_{ij})_{m,n}$ by a number k is a matrix $B=(b_{ij})_{m,n}$ whose elements b_{ij} are k times the elements of A, i.e. $b_{ij}=ka_{ij}$, $i=\overline{1,m}$, $j=\overline{1,n}$.

Note. The subtraction can be defined then in the following way

$$C=A-B=A+(-1)B.$$

Example. $3(3 \quad -1 \quad 2)=(3\cdot 3 \quad 3\cdot(-1) \quad 3\cdot 2)=(9 \quad -3 \quad 6)$.

Basic properties of these operations:

1. $A+B=B+A$ (commutability)
2. $(A+B)+C=A+(B+C)$ (associability)
3. $(ks)A=k(sA)=s(kA)$
4. $(k+s)A=kA+sA$
5. $k(A+B)=kA+kB$

Definition. The matrix A^T obtained from the given matrix A by interchanging its rows with its columns is called the transposed matrix.

Definition. The matrix A is called the symmetrical matrix if $A=A^T$, i.e.

$$a_{ij}=a_{ji} \quad i=\overline{1,m},\ j=\overline{1,n}.$$

Definition. The matrix A is called skew-symmetric matrix if $A=-A^T$, i.e.

$$a_{ij}=-a_{ji} \quad i=\overline{1,m},\ j=\overline{1,n}.$$

The definition of matrix multiplication is such that two matrices A and B can only be multiplied together to form their product AB when the number of columns of A is equal to the number of rows of B.

Definition. Product of the matrix A of the size m by n and the matrix B of the size n by p is the matrix C=AB of the size m by p with elements defined as

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\ldots+a_{in}b_{nj},\ i=\overline{1,m},\ j=\overline{1,p}.$$

Note. To calculate the element c_{ij} of the matrix C=AB we use all elements of the first matrix from the row number i and all elements of the second matrix from the column number j. That is why the rule of matrix multiplication can be called the rule "row by column" and formulated in the following way:

In order to get the element of the matrix C=AB situated in the i^{th} row and j^{th} column it is necessary to multiply all elements of the matrix A from i^{th} row by the corresponding elements of the matrix B from j^{th} column and then summarize the obtained products.

Basic properties of the matrix multiplication:

1. $k(AB)=(kA)B=A(kB)$
2. $(A+B)C=AC+BC$, $C(A+B)=CA+CB$ (distributivity)
3. $(AB)C=A(BC)$ (associability)
4. $AE=A$, $EA=A$
5. $(AB)^{T}=B^{T}A^{T}$

In general case $AB\neq BA$. Moreover, both AB and BA do not always exist at the same time.

Definition. The matrixes A and B that satisfy the equality AB=BA are called the commute (or commuting or commutative) matrices.

Definition. The matrix A to the power n is the matrix $C=\underbrace{A\cdot A\cdot A\cdot A\cdot\ldots\cdot A}_{n\text{ times}}$.

Example. Let us find the n-th power of the matrix A, where

$$A=\begin{pmatrix}0&2&0\\0&0&3\\0&0&0\end{pmatrix}.$$

First, we are going to find several first powers of A to understand the rule.

$$A^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0\cdot 0+2\cdot 0+0\cdot 0 & 0\cdot 2+2\cdot 0+0\cdot 0 & 0\cdot 0+2\cdot 3+0\cdot 0 \\ 0\cdot 0+0\cdot 0+3\cdot 0 & 0\cdot 2+0\cdot 0+3\cdot 0 & 0\cdot 0+0\cdot 3+3\cdot 0 \\ 0\cdot 0+0\cdot 0+0\cdot 0 & 0\cdot 2+0\cdot 0+0\cdot 0 & 0\cdot 0+0\cdot 3+0\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$A^3 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0\cdot 0+2\cdot 0+6\cdot 0 & 0\cdot 2+0\cdot 0+6\cdot 0 & 0\cdot 0+0\cdot 3+6\cdot 0 \\ 0\cdot 0+0\cdot 0+0\cdot 0 & 0\cdot 2+0\cdot 0+0\cdot 0 & 0\cdot 0+0\cdot 3+0\cdot 0 \\ 0\cdot 0+0\cdot 0+0\cdot 0 & 0\cdot 2+0\cdot 0+0\cdot 0 & 0\cdot 0+0\cdot 3+0\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since for any n such that n>3 $A^n = A^{n-3}A^3$ and A^3 is zero matrix then $A^n = 0$ for n>3.

1.3. The Determinants of the Second and the Third Orders

Any square matrix A of the n^{th} order can be associated with some value (number) called a determinant of the n^{th} order and signed as det$A$ or $|A|$.

Definition. Determinant of the second order $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is a number corresponding to the square matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and calculated by the formula:

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example. $\begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} = 1\cdot 3 - 2\cdot 5 = 3 - 10 = -7$.

Definition. Determinant of the third order $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is a number corresponding to the square matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and calculated by the formula:

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}$$

.

To remember this formula use the following "rule of triangles": elements-vertices of each triangle are factors in triples from the formula. Take the product with sign "+" if the corresponding triangle has side parallel to main diagonal of matrix and with sign "–" if it has side parallel to the secondary diagonal (Fig. 1). Notice that 2 triangles are singular.

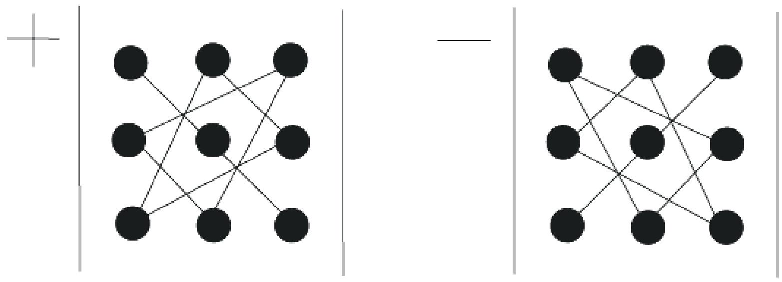


Figure 1. "Rule by triangles" to calculate the determinant of the third order.

Example.

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 5 \\ 1 & -1 & 0 \end{vmatrix}=1\cdot 3\cdot 0+2\cdot 5\cdot 1+(-1)\cdot 2\cdot(-1)-(-1)\cdot 3\cdot 1-2\cdot 2\cdot 0-1\cdot 5\cdot(-1)=$$

$$=0+10+2+3-0+5=20.$$

1.4. The Determinant of the n^{th} Order. Preliminary Idea

The determinant of the second order is the sum of two factorial terms where each one is the product of two elements of matrix taken by one from each row and each column with sign "+" or "–".The determinant of the third order is the sum of <u>three factorial</u> terms where each one is the product of <u>three</u> elements of matrix taken by one from each row and each column with sign "+" or "–".

It is natural to assume that the determinant of <u>the n^{th}</u> order is the sum of <u>n factorial</u> terms where each one is the product of <u>n</u> elements of matrix taken by one from each row and each column with sign "+" or "–". The only question is how to define sign for each product? To answer this question we introduce several new concepts.

1.5. Permutations and Substitutions

Definition. Permutation of numbers 1,2,3,…*n* (or any different numbers $a_1, a_2, a_3, \ldots a_n$) is any arrangement of them in some order.

Example. (1 2 3 4), (1 3 4 2), (2 3 1 4) are several different permutations of numbers 1,2,3,4.

Note. Since any *n* numbers may be numbered by numbers 1,2,3,…*n* it is enough to investigate the properties of permutations of numbers 1,2,3,…*n*.

Number of all possible permutations is equal to 1*2*3*…**n*=*n*!.

So, for example, number of all permutations of numbers 1,2,3,4 is 4!=24.

Definition. Two numbers *i* and *j* in the permutation are said to create *an inversion* if *i*>*j* and *i* is followed by *j*, i.e. the bigger number stands before the smaller one.

Number of all inversions in the permutation $(a_1\ a_2\ a_3\ \ldots\ a_n)$ we denote as $N(a_1\ a_2\ a_3\ \ldots\ a_n)$ or $[a_1\ a_2\ a_3\ \ldots\ a_n]$.

Example. Let us calculate number of inversions in the permutation (3 1 4 2). Number 3 forms 2 inversions with numbers 1 and 2 (they are less than 3 but situated after 3); number 1 forms 0 inversions with numbers 4 and 2 since 1 is less than 4 and 2; number 4 forms 1 inversion with number 2; number 2 forms 0 inversions since it is the last number. So, [3 1 4 2]=2+0+1+0=3.

Definition. The permutation is called odd if the number of all its inversions is odd number. And this permutation is called even if the number of all its inversions is even number.

If you change 2 elements in a permutation by their places then you obtain new permutation. This action is called *transposition*.

Theorem (about parity of permutation) Any transposition of permutation changes its parity, i.e. odd permutation becomes even and vice versa.

Proof. Let us consider the transposition of the elements i and j.

Suppose for the beginning that these elements stand side by side. After transposition all inversions with static elements will be saved since relational position of these elements with respect i and j will be the same, i.e. all smaller numbers from the right will stay to the right and all bigger numbers from the left will stay to the left of numbers i and j. So the only inversion could be changed is inversion between numbers i and j. If they formed an inversion before transposition, then they would not form it after and number of all transposition would be less by 1. If they did not then new inversion would appear after transposition. But in any case parity of transposition would be changed.

Suppose now that the elements i and j have s elements between them, i.e. the permutation looks like

$$(\ldots i\ k_1\ k_2\ k_3 \ldots k_s\ \ j \ldots) \text{ or } (\ldots j\ k_1\ k_2\ k_3 \ldots k_s\ i \ldots).$$

Let us consider the first variant. To change i and j by their places we should make s transposition of element i with neighbour elements $k_1, k_2, k_3, \ldots k_s$, one transposition of i and j and s transposition of j with neighbour elements $k_1, k_2, k_3, \ldots k_s$, i.e. we should make $2s+1$ transpositions of neighbour elements. At that the parity of the initial permutation changes odd number times, i.e. the parity of new permutation is different from the parity of initial one.

Similarly we can prove this statement for the permutation $(\ldots j\ k_1\ k_2\ k_3 \ldots k_s\ i \ldots)$. ***Theorem is proven.***

Definition. One-to-one mapping F of the set of the first n natural numbers $1,2,3,\ldots n$ into itself is called the substitution of the n^{th} order and written as

$$F = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ a_{i_1} & a_{i_2} & \cdots & a_{i_n} \end{pmatrix},$$

where the first row is the permutation of the numbers $1,2,3,\ldots n$ and the second row is the permutation of their images.

***Note* 1.** There are $n!$ different recording of the same substitution corresponding to $n!$ different permutations of numbers $1,2,3,\ldots n$ in the first row. New one can be obtained from another by transposition of the columns.

***Note* 2.** There are $n!$ different substitutions of the n^{th} order. They correspond to $n!$ different permutations of numbers $1,2,3,\ldots n$ in the second row when the first row is fixed.

Example. Let us consider the following one-to-one mapping:

F: 1 ←→ 2
2 ←→ 4
3 ←→ 3
4 ←→ 1
5 ←→ 5

It can be written, for example, as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 1 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 5 & 3 & 4 & 1 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}.$$

Definition. The number of inversions of substitution is equal to the number of inversions in the upper permutation and in the lower permutation.

Let us denote it as $N\begin{pmatrix} k_1 & k_2 & k_3 & \ldots & k_n \\ s_1 & s_2 & s_3 & \cdots & s_n \end{pmatrix}$.

Then

$$N\begin{pmatrix} k_1 & k_2 & k_3 & \ldots & k_n \\ s_1 & s_2 & s_3 & \cdots & s_n \end{pmatrix} = [k_1 \quad k_2 \quad k_3 \quad \ldots \quad k_n] + [s_1 \quad s_2 \quad s_3 \quad \ldots \quad s_n].$$

Note. This number depends on the record of the substitution.

Theorem (about parity of substitution) Parity of the number of inversions of substitution does not depend on its record.

Proof. New record can be obtained from another by transposition of the substitution columns. But at that you change two elements in the upper permutation and two elements in the lower permutation. It follows from Theorem 1 that parity of each permutation changes, i.e. the parity of sum is the same. ***Theorem is proven.***

Corollary. From the last two Theorems it follows that

$$(-1)^{[\ldots i \ldots j \ldots]} = -(-1)^{[\ldots j \ldots i \ldots]},$$

$$(-1)^{N\begin{pmatrix} \ldots & k_i & \cdots & k_j & \cdots \\ \ldots & s_i & \cdots & s_j & \cdots \end{pmatrix}} = (-1)^{N\begin{pmatrix} \ldots & k_j & \cdots & k_i & \cdots \\ \ldots & s_j & \cdots & s_i & \cdots \end{pmatrix}}.$$

1.6. Definition of the Determinant of the n^{th} Order

Consider an arbitrary square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \ddots & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$$

Definition. The determinant of the n^{th} order of the square matrix A is the number

$$\det A = \sum (-1)^N a_{s_1k_1} a_{s_2k_2} a_{s_3k_3} \ldots a_{s_nk_n},$$

where N is the number of inversions of the substitution $\begin{pmatrix} s_1 & s_2 & \ldots & s_n \\ k_1 & k_2 & \ldots & k_n \end{pmatrix}$ and summarizing is completed by all possible substitutions of the n^{th} order.

Note 1. In every product we use elements given by one from each row and each column.

Note 2. Number of terms in the sum is equal to $n!$.

Note 3. We can change the order of elements in each product in the order of increasing the first or the second subscripts. Then we get new formulas to calculate the determinant of the n^{th} order:

$$\det A = \sum (-1)^{N\begin{pmatrix} s_1 & s_2 & \dots & s_n \\ k_1 & k_2 & \dots & k_n \end{pmatrix}} a_{s_1k_1} a_{s_2k_2} \dots a_{s_nk_n} =$$
$$= \sum (-1)^{N\begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}} a_{1j_1} a_{2j_2} \dots a_{nj_n} =$$
$$= \sum (-1)^{[1 \; 2 \; \dots \; n]+[j_1 \; j_2 \; \dots \; j_n]} a_{1j_1} a_{2j_2} \dots a_{nj_n} = ;$$
$$= \sum (-1)^{[j_1 \; j_2 \; \dots \; j_n]} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

$$\det A = \sum (-1)^{N\begin{pmatrix} s_1 & s_2 & \dots & s_n \\ k_1 & k_2 & \dots & k_n \end{pmatrix}} a_{s_1k_1} a_{s_2k_2} \dots a_{s_nk_n} =$$
$$= \sum (-1)^{N\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ 1 & 2 & \dots & n \end{pmatrix}} a_{i_11} a_{i_22} \dots a_{i_nn} =$$
$$= \sum (-1)^{[i_1 \; i_2 \; \dots \; i_n]+[1 \; 2 \; \dots \; n]} a_{i_11} a_{i_22} \dots a_{i_nn} =$$
$$= \sum (-1)^{[i_1 \; i_2 \; \dots \; i_n]} a_{i_11} a_{i_22} \dots a_{i_nn}$$

So we get

$$\det A = \sum (-1)^{[j_1 \; j_2 \; \dots \; j_n]} a_{1j_1} a_{2j_2} a_{3j_3} \dots a_{nj_n} ,$$
$$\det A = \sum (-1)^{[i_1 \; i_2 \; \dots \; i_n]} a_{i_11} a_{i_22} a_{i_33} \dots a_{i_nn} .$$

Example. By means of the definition of the determinant find the formula of the determinant of the third order.

Let us keep the first subscripts the same and consider all possible permutations of the second subscripts, i.e. we are going to use the second given above formula to calculate the determinant.

Permutation $(j_1\, j_2\, j_3)$	Number of inversions $[j_1\, j_2\, j_3]$	Sign of $(-1)^{[j_1 j_2 j_3]}$	Product $a_{1j_1} a_{2j_2} a_{3j_3}$
(1 2 3)	0+0+0=0	+	$a_{11}a_{22}a_{33}$
(1 3 2)	0+1+0=1	–	$a_{11}a_{23}a_{32}$
(2 1 3)	1+0+0=1	–	$a_{12}a_{21}a_{33}$
(2 3 1)	1+1+0=2	+	$a_{12}a_{23}a_{31}$
(3 1 2)	2+0+0=2	+	$a_{13}a_{21}a_{32}$
(3 2 1)	2+1+0=3	–	$a_{13}a_{22}a_{31}$

Thus

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

So, we obtained the same expression as the given one in Section 1.3.

1.7. Properties of the Determinant of the n^{th} Order

Property 1. $\det A = \det A^T$.

Proof. $\det A = \sum (-1)^{[j_1 \; j_2 \; \cdots \; j_n]} a_{1j_1} a_{2j_2} a_{3j_3} \dots a_{nj_n}$. Let us denote elements of A^T as a'_{ij}, $i = \overline{1,n}$, $j = \overline{1,n}$. Then $a'_{ij} = a_{ji}$.

By definition:

$$\det A^T = \sum (-1)^{[j_1 \; j_2 \; \cdots \; j_n]} a'_{j_1 1} a'_{j_2 2} a'_{j_3 3} \dots a'_{j_n n} =$$
$$= \sum (-1)^{[j_1 \; j_2 \; \cdots \; j_n]} a_{1j_1} a_{2j_2} a_{3j_3} \dots a_{nj_n}.$$

We obtained the same expression as for $\det A$. ***The property is proven.***

Note. All properties valid for the rows are valid for the columns.

Property 2. The determinant of the matrix with zero-row or zero-column is equal to zero.

Proof. Let the i^{th} row be zero-row. Then

$$\det A = \sum (-1)^{[j_1 \; j_2 \; \cdots \; j_n]} a_{1j_1} \dots a_{ij_i} \dots a_{nj_n} =$$
$$= \sum (-1)^{[j_1 \; j_2 \; \cdots \; j_n]} a_{1j_1} \dots 0 \dots a_{nj_n} = \sum 0 = 0.$$

The property is proven.

Property 3. If the determinant Δ' is obtained from Δ by changing any two rows by their places then $\Delta' = -\Delta$, i.e. at changing two rows by their places the sign of the determinant changes by the opposite one.

Proof. Let us interchange the i^{th} and the j^{th} rows. If the product $a_{1k_1} a_{2k_2} a_{3k_3} \dots a_{nk_n}$ is term of the determinant Δ then it is the term of Δ' as well, and *vice versa*. So, these determinants have the same terms. The only difference can be in sign. But the signs of term $a_{1k_1}\dots a_{ik_i}\dots a_{jk_j}\dots a_{s_nk_n}$ in Δ and Δ' depend on parities of $N\begin{pmatrix} 1 & 2 & \dots & i & \dots & j & \dots & n \\ k_1 & k_2 & \dots & k_i & \dots & k_j & \dots & k_n \end{pmatrix}$, $N\begin{pmatrix} 1 & 2 & \dots & j & \dots & i & \dots & n \\ k_1 & k_2 & \dots & k_i & \dots & k_j & \dots & k_n \end{pmatrix}$, correspondingly, which are opposite by Theorem 1. It means that $\Delta' = -\Delta$. ***The property is proven.***

Property 4. The determinant with two identical rows (columns) is equal to 0.

Proof. If we interchange these two identical rows in the determinant Δ then new determinant Δ' will be at the same time equal to Δ and equal to $-\Delta$ by property 3.

So $\Delta' = \Delta = -\Delta \Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$. ***The property is proven.***

Property 5. If the determinants Δ and Δ' differ only in that the elements of some one row of Δ' are equal to k times corresponding elements of Δ then $\Delta' = k\Delta$, i.e. you can take the common factor k of the row (column) elements out the determinant.

Example.

$$\begin{vmatrix} 19 & -76 \\ 13 & 2 \end{vmatrix} = 19\begin{vmatrix} 1 & -4 \\ 13 & 2 \end{vmatrix} = 19\cdot 2\begin{vmatrix} 1 & -2 \\ 13 & 1 \end{vmatrix} = 38(1\cdot 1 - 13\cdot(-2)) = 38\cdot 27 = 1026.$$

Proof. Let the i^{th} row in Δ' have the elements $ka_{i1}, ka_{i2}, ka_{i3}, \dots ka_{in}$. Then

$$\Delta' = \sum (-1)^{[j_1 \;\; j_2 \;\; \cdots \;\; j_n]} a_{1j_1}\dots a_{i-1\,j_{i-1}}(ka_{ij_i})a_{i+1\,j_{i+1}}\dots a_{nj_n} =$$

$$= k\sum (-1)^{[j_1 \;\; j_2 \;\; \cdots \;\; j_n]} a_{1j_1}\dots a_{i-1\,j_{i-1}}a_{ij_i}a_{i-1\,j_{i-1}}\dots a_{nj_n} = k\Delta.$$

The property is proven.

Property 6. The determinant with two proportional rows (columns) is equal to zero.

Proof. Suppose the i^{th} and the j^{th} rows are proportional. Then

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ ka_{i1} & ka_{i2} & \dots & ka_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ (i^{th}\ row) \\ \\ (j^{th}\ row) \\ \\ \\ \end{matrix} = [\text{by property } 5] =$$

$$= k\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ (i^{th}\ row) \\ \\ (j^{th}\ row) \\ \\ \\ \end{matrix} = [\text{by property } 4] = 0.$$

The property is proven.

Property 7. Suppose each element of the i^{th} row in the determinant Δ is equal to sum $a_{ij} = c_{ij} + b_{ij}\ (j = \overline{1,n})$. Then $\Delta = \Delta' + \Delta''$, where Δ' is obtained from Δ by replacing its elements from i^{th} row a_{ij} by c_{ij} and Δ'' is obtained from Δ by replacing its elements from i^{th} row a_{ij} by b_{ij}.

Proof.

$$\Delta = \sum (-1)^{[j_1\ \ j_2\ \ \cdots\ \ j_n]} a_{1j_1} \dots a_{i-1\,j_{i-1}} (c_{ij_i} + b_{ij_i}) a_{i+1\,j_{i+1}} \dots a_{nj_n} =$$

$$= \sum (-1)^{[j_1\ \ j_2\ \ \cdots\ \ j_n]} (a_{1j_1} \dots a_{i-1\,j_{i-1}} c_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n} + a_{1j_1} \dots a_{i-1\,j_{i-1}} b_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n}) =$$

$$= \sum (-1)^{[j_1\ \ j_2\ \ \cdots\ \ j_n]} a_{1j_1} \dots a_{i-1\,j_{i-1}} c_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n} +$$

$$+ \sum (-1)^{[j_1\ \ j_2\ \ \cdots\ \ j_n]} a_{1j_1} \dots a_{i-1\,j_{i-1}} b_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n} = \Delta' + \Delta''.$$

The property is proven.

Example.

$$\begin{vmatrix} 1 & \cos^2 x & \sin^2 x \\ 1 & \cos^2 y & \sin^2 y \\ 1 & \cos^2 z & \sin^2 z \end{vmatrix} = \begin{vmatrix} \cos^2 x + \sin^2 x & \cos^2 x & \sin^2 x \\ \cos^2 y + \sin^2 y & \cos^2 y & \sin^2 y \\ \cos^2 z + \sin^2 z & \cos^2 z & \sin^2 z \end{vmatrix} =$$

$$= \begin{vmatrix} \cos^2 x & \cos^2 x & \sin^2 x \\ \cos^2 y & \cos^2 y & \sin^2 y \\ \cos^2 z & \cos^2 z & \sin^2 z \end{vmatrix} + \begin{vmatrix} \sin^2 x & \cos^2 x & \sin^2 x \\ \sin^2 y & \cos^2 y & \sin^2 y \\ \sin^2 z & \cos^2 z & \sin^2 z \end{vmatrix} = [\text{by pr.4}] = 0 + 0 = 0.$$

Property 8. The value of the determinant does not change if one adds to the elements of one row of this determinant the corresponding elements of another row multiplied by some number.

Proof.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{j1} + ka_{i1} & a_{j2} + ka_{i2} & \dots & a_{jn} + ka_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{array}{l} \\ \\ (i^{th}) \\ \\ (j^{th}) \\ \\ \\ \end{array} = [\text{by pr.5,7}] = k \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = [\text{by pr.4}] = 0 + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

The property is proven.

Property 9. If $a_{is}=\sum_{\substack{j=1\\ j\neq i}}^{n}k_j a_{js}\ (s=\overline{1,n})$ in the determinant then it is equal to zero, i.e. if one of the row (column) in the determinant is a linear combination of other rows (columns) then this determinant is equal to zero.

Proof. Without loss of generality we can assume that

$$a_{1s}=\sum_{j=2}^{n}k_j a_{js}\ (s=\overline{1,n}).$$

Then

$$\begin{vmatrix} \sum_{j=2}^{n}k_j a_{j1} & \sum_{j=2}^{n}k_j a_{j2} & \cdots & \sum_{j=2}^{n}k_j a_{jn} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = [\text{by pr.5,7}] = \sum_{j=2}^{n}k_j \begin{vmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} =$$

$= [\text{by pr.4}] = \sum_{j=2}^{n}k_j 0 = 0$. ***The property is proven.***

Example. In the example to property 7 $a_{11}=1\cdot a_{12}+1\cdot a_{13}$, $a_{21}=1\cdot a_{22}+1\cdot a_{23}$, $a_{31}=1\cdot a_{32}+1\cdot a_{33}$, i.e. the first column is linear combination (in this case just sum) of the others columns. By property 9 this determinant is equal to zero.

Property 10. Let A and B be square matrices of the n^{th} order. Then

$$\det AB=\det A\det B.$$

Without proof.

1.8. Algebraic Cofactors and Minors

Definition. The minor M_{ij} of the element a_{ij} of the determinant of the n^{th} order is the determinant of the $(n\text{-}1)^{th}$ order obtained by removing from the initial determinant the i^{th} row and the j^{th} column, i.e. the row and the column of the element a_{ij}.

Definition. The algebraic cofactor (or just cofactor) A_{ij} of the element a_{ij} of the determinant of the n^{th} order is a number $(-1)^{i+j}M_{ij}$.

Example. For the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 2 & 0 \end{vmatrix}$$

we have

$$M_{12} = \begin{vmatrix} 4 & 2 & 1 \\ 0 & 0 & 1 \\ -3 & 2 & 0 \end{vmatrix} = 4\cdot 0\cdot 0 + 2\cdot 1\cdot(-3) + 1\cdot 0\cdot 2 - 1\cdot 0\cdot(-3) - 2\cdot 0\cdot 0 - 4\cdot 1\cdot 2 = -14,$$

$$A_{12} = (-1)^{1+2}M_{12} = (-1)\cdot(-14) = 14.$$

Theorem (about the determinant with the only nonzero element in some row) If the determinant has a special form where all elements except the element a_{ij} in some row are equal to zero then this determinant is equal to the product of this element by its algebraic cofactor, i.e.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ & & & \vdots & & \\ 0 & 0 & \dots & a_{ij} & \dots & 0 \\ & & & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = a_{ij}(-1)^{i+j}M_{ij} = a_{ij}A_{ij}.$$

Proof. Let us consider first the determinant of the following form:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix} = \sum (-1)^{[k_1 \;\; k_2 \;\; \dots \;\; k_{n-1} \;\; k_n]} a_{1k_1}a_{2k_2}\dots a_{n-1k_{n-1}}a_{nk_n} =$$

=[Since the only non-zero element in the n^{th} row is at $k_n = n$]=

$$= \sum (-1)^{[k_1 \;\; k_2 \;\; \dots \;\; k_{n-1} \;\; n]} a_{1k_1}a_{2k_2}\dots a_{n-1k_{n-1}}a_{nn} =$$

$$= a_{nn}\sum(-1)^{[k_1 \;\; k_2 \;\; \dots \;\; k_{n-1} \;\; n]} a_{1k_1}a_{2k_2}\dots a_{n-1k_{n-1}} =$$

$$= a_{nn}\sum(-1)^{[k_1 \;\; k_2 \;\; \dots \;\; k_{n-1}]} a_{1k_1}a_{2k_2}\dots a_{n-1k_{n-1}} =$$

$$= a_{nn}M_{nn} = a_{nn}(-1)^{n+n}M_{nn} = a_{nn}A_{nn}.$$

So, for this case the theorem is proven.

Let us consider an arbitrary determinant satisfying the mentioned above condition. To put non-zero element a_{ij} on the place n,n we should make (*n*-*i*) interchange of the i^{th} row with all lower rows and (*n*-*j*) interchange of the j^{th} column with all columns to the right:

$$\begin{array}{r} \\ \\ \downarrow \\ (n-i) \\ \downarrow \\ \\ \end{array}\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ & & & \vdots & & \\ 0 & 0 & \dots & a_{ij} & \dots & 0 \\ & & & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}.$$
$$\rightarrow \quad (n-j) \quad \rightarrow$$

After these actions the determinant of the $(n\text{-}1)^{th}$ order in the upper left corner will be exactly the minor of the element a_{ij}. And accordingly to the proven before case, the obtained determinant Δ' will be equal to $a_{ij}M_{ij}$.

So, $\Delta = (-1)^{n-i}(-1)^{n-j}\Delta' = (-1)^{2n-i-j}\Delta' = (-1)^{i+j}\Delta' = (-1)^{i+j}a_{ij}M_{ij} = a_{ij}A_{ij}$.

Theorem is proven.

Theorem (Expansion of the determinant by cofactors) The value of the determinant is equal to the sum of products of the elements of any row (column) by their cofactors. The determinant does not depend on the choice of the row (column).

Proof. Let us consider the determinant of the n^{th} order and present all elements of k^{th} row as sum of (*n*-1) zeros and the element:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ a_{k1}+0+\dots+0 & 0+a_{k2}+0+\dots+0 & 0+0+a_{k3}+0+\dots+0 & \dots & 0+0+\dots+0+a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} =$$

=[by property 7]=

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ a_{k1} & 0 & 0 & \dots & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ 0 & a_{k2} & 0 & \dots & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} + \dots + \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} =$$

=[by the previous Theorem]= $a_{k1}A_{k1} + a_{k2}A_{k2} + a_{k3}A_{k3} + \dots + a_{kn}A_{kn}$. ***Theorem is proven.***

Note 1. Described in the theorem way to calculate the determinant could be called also as *Expansion of the determinant along the row* or *Expansion of the determinant down the column*.

Note 2. It follows from the Theorem that to calculate the determinant of the n^{th} order we should calculate n determinants of the $(n\text{-}1)^{\text{th}}$ order:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = a_{k1}A_{k1} + a_{k2}A_{k2} + a_{k3}A_{k3} + \dots + a_{kn}A_{kn} = \sum_{i=1}^{n} a_{ki}A_{ki};$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = a_{1s}A_{1s} + a_{2s}A_{2s} + a_{3s}A_{3s} + \dots + a_{ns}A_{ns} = \sum_{i=1}^{n} a_{is}A_{is}.$$

Since we can choose any row (column), to simplify calculations it will be better to choose row (column) with maximum number of zero elements.

To increase the number of zeros in the row (column) use property 8 of the determinants.

Example. Calculate the determinant:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -3 & 0 & 2 & 0 \end{vmatrix} =$$

=[Let us add to the elements of the first column the corresponding elements of the forth one multiplied by (−2) and to the elements of the second column the corresponding elements of the forth one multiplied by (−1)]=

$$= \begin{vmatrix} 1-8 & 2-4 & 3 & 4 \\ 4-2 & 3-1 & 2 & 1 \\ 2-2 & 1-1 & 0 & 1 \\ -3-0 & 0-0 & 2 & 0 \end{vmatrix} = \begin{vmatrix} -7 & -2 & 3 & 4 \\ 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 2 & 0 \end{vmatrix} =$$

=[Let us expand it along the third row]=

$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34} = 0 + 0 + 0 + 1 \cdot (-1)^{3+4} \begin{vmatrix} -7 & -2 & 3 \\ 2 & 2 & 2 \\ -3 & 0 & 2 \end{vmatrix} =$$

=[Let us add to the elements of the first row the elements of the second and use property 5]=

$$= -2 \begin{vmatrix} -5 & 0 & 5 \\ 1 & 1 & 1 \\ -3 & 0 & 2 \end{vmatrix} = 10 \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ -3 & 0 & 2 \end{vmatrix} = 10(a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}) =$$

$$= 10 \cdot 1 \cdot (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = 10(2-3) = -10.$$

Theorem (about the sum of products of the elements of any row/column by cofactors of the elements from another row/column) The sum of products of the elements of any row (column) by the cofactors of the corresponding elements of other row (column) is equal to zero.

Proof. Let us consider the sum of products of the elements of the i^{th} row by cofactors of the j^{th} row:

$$a_{i1}A_{j1}+a_{i2}A_{j2}+a_{i3}A_{j3}+\ldots+a_{in}A_{jn}=\left[\text{By previous Theorem}\right]=$$

$$=\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ (i^{th}\ row) \\ \\ (j^{th} row) \\ \\ \\ \end{matrix} =[\text{by } \ \text{pr.4}\ ]=0$$

Theorem is proven.

1.9. Rule by Cramer

Let us consider the system of linear algebraic equations (SLAE) with *n* equations and *n* unknowns:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2 \\ \vdots \\ a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n=b_n \end{cases}$$

and introduce next notations:

$\Delta=\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}$ (It is a determinant of the system consisting of the coefficients of unknowns);

$$\Delta_1=\begin{vmatrix} b_1 & a_{12} & a_{13} & \ldots & a_{1n} \\ b_2 & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & & \\ b_n & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}=b_1A_{11}+b_2A_{21}+\ldots+b_nA_{n1}=\sum_{i=1}^{n} b_iA_{i1};$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} & \dots & a_{1n} \\ a_{21} & b_2 & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & b_n & a_{n3} & \dots & a_{nn} \end{vmatrix} = b_1A_{12} + b_2A_{22} + \dots + b_nA_{n2} = \sum_{i=1}^{n} b_iA_{i2};$$

…

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & b_2 \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & b_n \end{vmatrix} = b_1A_{1n} + b_2A_{2n} + \dots + b_nA_{nn} = \sum_{i=1}^{n} b_iA_{in}.$$

Let us multiply the first equation of the given above system by A_{11}, the second equation by A_{21}, the third by A_{31}, … the n^{th} by A_{n1} and summarize these products collecting similar terms:

$$x_1(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \dots + a_{n1}A_{n1}) +$$
$$x_2(a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} + \dots + a_{n2}A_{n1}) +$$
$$x_3(a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31} + \dots + a_{n3}A_{n1}) +$$
$$\dots$$
$$+ x_n(a_{1n}A_{11} + a_{2n}A_{21} + a_{3n}A_{31} + \dots + a_{nn}A_{n1}) =$$
$$= b_1A_{11} + b_2A_{21} + \dots + b_nA_{n1}.$$

By two last Theorems of the previous section we have from above:

$$x_1\Delta + x_2 0 + x_3 0 + \dots + x_n 0 = \Delta_1 \Leftrightarrow x_1\Delta = \Delta_1 \Rightarrow \left(\text{if } \Delta \neq 0 \text{ then } x_1 = \frac{\Delta_1}{\Delta}\right).$$

In the same way if one multiplies the first equation of the system by A_{1k}, the second equation by A_{2k}, the third by A_{3k}, … the n^{th} by A_{nk} one has:

$$x_1 0 + \dots + x_{k-1} 0 + x_k\Delta + x_{k+1} 0 + \dots + x_n 0 = \Delta_k \Leftrightarrow x_k\Delta = \Delta_k \Rightarrow$$

$$\Rightarrow \text{if } \Delta \neq 0 \text{ then } x_k = \frac{\Delta_k}{\Delta}.$$

So we proved the following theorem:

Theorem (Rule by Cramer) If the determinant of the SLAE with n equations and n variables is not equal to zero then this SLAE has the only solution evaluated by formulas:

$$x_k = \frac{\Delta_k}{\Delta},\ k = \overline{1,n}.$$

Example. Solve the system $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + x_2 - 2x_3 = 5 \\ x_1 - x_2 - 4x_3 = 5 \end{cases}$.

Let us find the determinant of the system Δ:

$$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & -2 \\ 1 & -1 & -4 \end{vmatrix} = -4 - 4 + 0 - 0 + 24 - 2 = 14.$$

Since the determinant is not equal to zero we can use the rule by Cramer:

$$\Delta_1 = \begin{vmatrix} 1 & 2 & 0 \\ 5 & 1 & -2 \\ 5 & -1 & -4 \end{vmatrix} = -4 - 20 + 0 - 0 + 40 - 2 = 14,$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 5 & -2 \\ 1 & 5 & -4 \end{vmatrix} = -20 - 2 - 0 + 0 + 12 + 10 = 0,$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ 1 & -1 & 5 \end{vmatrix} = 5 + 10 - 3 - 1 - 30 + 5 = -14.$$

So, $x_1 = \frac{\Delta_1}{\Delta} = \frac{14}{14} = 1,\ x_2 = \frac{\Delta_2}{\Delta} = \frac{0}{14} = 0,\ x_3 = \frac{\Delta_3}{\Delta} = \frac{-14}{14} = -1.$

1.10. Inverse Matrix

Definition. Square matrix A is called a non-singular matrix if $\det A \neq 0$. In other case it is called a singular.

Definition. Matrix A^{-1} is called an inverse matrix to the square matrix A if $A^{-1}A = AA^{-1} = E$.

Example. Suppose $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$,

$$B = \begin{pmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2 t + \sin^2 t & \cos t \sin t - \sin t \cos t \\ \sin t \cos t - \cos t \sin t & \sin^2 t + \cos^2 t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2 t + \sin^2 t & -\cos t \sin t + \sin t \cos t \\ -\sin t \cos t + \cos t \sin t & \sin^2 t + \cos^2 t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $B = A^{-1}$.

Definition. Matrix $A^{cof} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$ is called the cofactor matrix of the square matrix A.

Definition. Matrix $A^{ad} = (A^{cof})^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$ is called the adjoint matrix of the matrix A.

Theorem (Necessary condition for the matrix to have an inverse matrix)

If matrix A has an inverse matrix A^{-1} then $\det A \neq 0$.

Proof. Let us prove the theorem from the contrary. Let $\det A = 0$. By property 10 of the determinants:

$1 = \det E = \det(AA^{-1}) = \det A \det A^{-1} = 0 \det A^{-1} = 0$. We got contradiction.

Theorem is proven.

Theorem (Sufficient condition for the matrix to have an inverse matrix)

Any non-singular matrix *A* has an inverse matrix and only one.

Proof. Let us prove that matrix $B = \dfrac{1}{\det A} A^{ad}$ is inverse matrix to the matrix *A*.

$$AB = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix} =$$

$$\frac{1}{\det A} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix} =$$

=[by Theorem on determinant decomposition and Theorem about sum of products of row/column elements by algebraic cofactors of elements from another row/columns]=

$$= \frac{1}{\det A} \begin{pmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \det A \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix} = E.$$

In the same way it can be proven that $BA = E$.

It means that *B* is inverse matrix of *A*. Let us prove that *A* does not have another inverse matrices.

Suppose C is another inverse matrix of *A*, i.e. $AB = BA = E$ and $CA = AC = E$. Then $C = CE = CAB = EB = B$.

It means that *B* is the only inverse matrix. ***Theorem is proven.***

So

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix} \text{ if } \det A \neq 0.$$

By means of the obtained above formula for inverse matrix It is simple to prove the following properties of the inverse matrices:

1. $\left(A^{-1}\right)^{-1} = A$;
2. $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$;
3. $(AB)^{-1} = B^{-1}A^{-1}$;
4. $\det A^{-1} = (\det A)^{-1}$.

1.11. Solving the Matrix Equations by Means of Inverse Matrix

Let us consider three types of matrix equations.

Type 1. $AX = B$, where A is square non-singular matrix.

By the second theorem of the previous section A has an inverse matrix A^{-1}. Let us multiply the equation by A^{-1} from the left. Then

$$AX = B \Leftrightarrow A^{-1}AX = A^{-1}B \Leftrightarrow EX = A^{-1}B \Leftrightarrow X = A^{-1}B.$$

Type 2. $XA = B$, where A is square non-singular matrix.

By the second theorem of the previous section A has an inverse matrix A^{-1}. Let us multiply the equation by A^{-1} from the right. Then

$$XA = B \Leftrightarrow XAA^{-1} = BA^{-1} \Leftrightarrow XE = BA^{-1} \Leftrightarrow X = BA^{-1}.$$

Type 3. $AXC = B$, where A and C are square non-singular matrix.

By the second theorem of the previous section A and C have the inverse matrices. Let us multiply the equation by A^{-1} from the left and by C^{-1} from the right. Then

$$AXC = B \Leftrightarrow A^{-1}AXCC^{-1} = A^{-1}BC^{-1} \Leftrightarrow X = A^{-1}BC^{-1}.$$

Example. Solve the system

$$\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + x_2 - 2x_3 = 5 \\ x_1 - x_2 - 4x_3 = 5 \end{cases}.$$

Let us introduce some matrices:

$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & -2 \\ 1 & -1 & -4 \end{pmatrix}$, i.e. the matrix of the system;

$B = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$, i.e. the column matrix of right sides;

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, i.e. the column matrix of the unknowns.

Then the initial system can be rewritten in the form

$$AX=B.$$

Since the determinant of the matrix A

$$\det A = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & -2 \\ 1 & -1 & -4 \end{vmatrix} = -4 - 4 + 0 - 0 + 24 - 2 = 14 \neq 0$$

we can use the inverse matrix A^{-1} to obtain the solution of the system (Type 1). Let us calculate all cofactors of the matrix A:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -1 & -4 \end{vmatrix} = -6, \quad A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 0 \\ -1 & -4 \end{vmatrix} = 8, \quad A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix} = -4;$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 1 & -4 \end{vmatrix} = 10, \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 1 & -4 \end{vmatrix} = -4, \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} = 2;$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4, \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = 3, \quad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5.$$

So

$$A^{-1} = \frac{1}{14}\begin{pmatrix} -6 & 8 & -4 \\ 10 & -4 & 2 \\ -4 & 3 & -5 \end{pmatrix};$$

$$X = A^{-1}B = \frac{1}{14}\begin{pmatrix} -6 & 8 & -4 \\ 10 & -4 & 2 \\ -4 & 3 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -6+40-20 \\ 10-20+10 \\ -4+15-25 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 14 \\ 0 \\ -14 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

i.e. $x_1 = 1$, $x_2 = 0$, $x_3 = -1$.

1.12. Concept of Linear Dependence

Let us denote each row of the matrix A of the size m by n as e_k $(k = \overline{1, m})$, i.e.

$e_1 = (a_{11} \quad a_{12} \quad \ldots \quad a_{1n})$,

$e_2 = (a_{21} \quad a_{22} \quad \ldots \quad a_{2n})$,

…

$e_k = (a_{k1} \quad a_{k2} \quad \ldots \quad a_{kn})$,

…

$e_n = (a_{n1} \quad a_{n2} \quad \ldots \quad a_{nn})$

and each column of this matrix as t_k $(k = \overline{1, n})$, i.e.

$$t_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix},\ t_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix},\ \ldots,\ t_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix},\ \ldots,\ t_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Definition. Expression $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_m e_m$, where α_k $(k = \overline{1, m})$ are real numbers, is called linear combination of rows. Expression $\gamma_1 t_1 + \gamma_2 t_2 + \gamma_3 t_3 + \ldots + \gamma_n t_n$, where γ_k $(k = \overline{1, m})$ are real numbers, is called linear combination of columns.

Definition. Rows of the matrix are linearly dependent (LD) if there is some zero linear combination of the rows with at least one nonzero coefficient, i.e.

$e_1, e_2, \ldots, e_m$ are LD $\Leftrightarrow \left(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_m e_m = 0 \quad \text{and} \quad \exists i_0 : \alpha_{i_0} \neq 0\right)$.

Definition. Rows of the matrix are linearly independent (LI) if any linear combination equal to zero has zero coefficients, i.e.

$$e_1, e_2, \ldots, e_m \text{ are LI} \Leftrightarrow \left(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_m e_m = 0 \quad \Rightarrow \alpha_i = 0 \ \forall i\right).$$

Note. The same definitions of linear dependence and linear independence can be introduced for columns.

Example. Let us consider matrix $A = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 9 & 12 \end{pmatrix}$. Here

$e_3 = \begin{pmatrix} 2 & 9 & 12 \end{pmatrix} = 2\begin{pmatrix} -1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} = 2e_1 + e_2$. Thus, $2e_1 + e_2 - e_3 = 0$. We have linear combination of the rows equal to zero. But $\alpha_1 = 2 \neq 0$, $\alpha_2 = 1 \neq 0$, $\alpha_3 = -1 \neq 0$. It means that rows of this matrix are LD.

Theorem (Criterion of linear dependence for the rows) For rows of the matrix to be linearly dependent it is necessary and sufficient that one of them is linear combination of other rows.

Proof. Necessity. We know that rows of the matrix are linearly dependent. We should proof that one of them is linear combination of other rows. Suppose we have some zero linear combination of rows. From definition of linear dependence we know that at least one coefficient is not equal to zero. Suppose it has number k, i.e. $\alpha_k \neq 0$. Let us divide zero expression by $-\alpha_k$ and express the row e_k from the obtained equation:

$$\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_k e_k + \ldots + \alpha_m e_m = 0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{\alpha_1}{\alpha_k} e_1 - \frac{\alpha_2}{\alpha_k} e_2 - \ldots - e_k - \ldots - \frac{\alpha_m}{\alpha_k} e_m = 0$$

$$\Leftrightarrow e_k = -\frac{\alpha_1}{\alpha_k} e_1 - \frac{\alpha_2}{\alpha_k} e_2 - \ldots - \frac{\alpha_{k-1}}{\alpha_k} e_{k-1} - \frac{\alpha_{k+1}}{\alpha_k} e_{k+1} - \ldots - \frac{\alpha_m}{\alpha_k} e_m = 0 \Leftrightarrow$$

$$\Leftrightarrow e_k = -\sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{\alpha_i}{\alpha_k} e_i .$$

Necessity is proven.

Sufficiency. Let $e_k = \sum_{\substack{i=1 \\ i \neq k}}^{m} \gamma_i e_i$. We should prove that rows are linearly dependent. Let us put e_k to the right of the last equation. So we have

$0 = \sum_{\substack{i=1 \\ i \neq k}}^{m} \gamma_i e_i - e_k$, i.e. zero linear combination of all rows with the coefficient $\gamma_k = -1 \neq 0$. From definition of linear dependence it follows that the rows are LD. ***Theorem is proven.***

1.13. Rank of the Matrix

Definition. Minor of the *k*-th order M_k of the matrix *A* of the size *m* by *n* ($0 \le k \le \min(m,n)$) is the determinant consisting of the elements standing in the intersection of any *k* rows and any *k* columns of the matrix *A*.

Example. $A = \begin{pmatrix} 1 & 2 & 4 & 0 & -3 \\ 2 & 3 & 1 & -1 & 0 \\ -8 & 6 & 5 & 7 & 9 \end{pmatrix}$.

The determinant $M_2 = \begin{vmatrix} 1 & 0 \\ -8 & 7 \end{vmatrix} = 37$ with elements from the first and the third rows and the first and the forth columns of *A* is one of the minors of the second order.

Definition. Rank of the matrix *A* is a maximum order of nontrivial (nonzero) minors of the matrix *A*. It is denoted as $rg(A)$ or $r(A)$.

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{pmatrix}$. The biggest order of the existing minor is 3.

$$M_3 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{vmatrix} = 0,\ M_2 = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0.$$

Thus, $rg(A) = 2$.

Definition. Suppose $r = r(A)$. Nonzero minor of r^{th} order is called the basic minor and rows and columns of the matrix *A* composing this minor are called the basic rows and columns.

Note. Sometimes there are several basic minors in the matrix *A*.

Theorem (about basic minor) The following statements are valid:

(i) Basic rows (columns) are linearly independent;

(ii) Any row (column) of matrix *A* is a linear combination of basic rows (columns).

Proof. (i): Let us assume that basic rows are linearly dependent. It means that one of the rows in the basic minor is linear combination of other rows. From property 9 of the determinants it follows that basic minor is equal to zero. We got contradiction with definition of the basic minor. Statement (i) is proven.

(ii): Without loss of generality we can assume that basic minor is situated in the upper left corner of the matrix *A*. So,

$$M_r = \begin{vmatrix} a_{11} & \dots & a_{1r} \\ \vdots & & \\ a_{r1} & \dots & a_{rr} \end{vmatrix} \neq 0, \text{ where } r = r(A).$$

Let us consider the following determinant obtained from M_r by adding the corresponding elements of k^{th} row and j^{th} column of *A*:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{rr} & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2r} & a_{2j} \\ \vdots & & & & \\ a_{r1} & a_{r2} & \dots & a_{rr} & a_{rj} \\ a_{k1} & a_{k2} & \dots & a_{kr} & a_{kj} \end{vmatrix}.$$

There are two situations:

1) $j \le r$ or $k \le r$. Then we have two identical rows or columns in the determinant, i.e. $\Delta = 0$.

2) $j > r$ and $k > r$. Then Δ is a minor of the $(r+1)^{th}$ order of A and equal to zero since r is maximum order of nonzero minors.

Thus, in any case $\Delta = 0$. Let us expand this determinant down the $(r+1)^{th}$ column:

$$0 = a_{1j}A_{1r+1} + a_{2j}A_{2r+1} + \ldots + a_{rj}A_{r\,r+1} + a_{kj}A_{r+1r+1} =$$
$$= a_{1j}A_{1r+1} + a_{2j}A_{2r+1} + \ldots + a_{rj}A_{r\,r+1} + a_{kj}M_r .$$

Since $M_r \ne 0$, It follows that

$$0 = a_{1j}\frac{A_{1r+1}}{M_r} + \ldots + a_{rj}\frac{A_{rr+1}}{M_r} + a_{kj} \Leftrightarrow$$
$$\Leftrightarrow a_{kj} = -a_{1j}\frac{A_{1r+1}}{M_r} - \ldots - a_{rj}\frac{A_{rr+1}}{M_r} = \gamma_1 a_{1j} + \gamma_2 a_{2j} + \ldots \gamma_r a_{rj},$$

where coefficients γ_i depend on the elements of the k^{th} row and does not depend on the elements of the j^{th} column. Thus, each element of the k^{th} row is linear combination of the corresponding elements of basic rows, i.e. the k^{th} row is linear combination of basic rows.

In similar way we can prove these statements for the columns. ***Theorem is proven.***

Note. It follows from the theorem that the rank of the matrix is equal to the maximum number of linearly independent rows (columns) of this matrix.

1.14. Elementary Row Operations and Column Operations

Elementary row operations and column operations are a simple set of matrix operations that can be used to reduce a matrix to the *row echelon form or column echelon form*.

Definition. A matrix (whether square or rectangular) is in the row echelon form if:

- The first nonzero element of each nonzero row occurs in a column to the right of the first nonzero element in the previous row.
- Rows that are completely zero occur last.

Definition. A matrix is in the *row reduced echelon form* if it is in the row echelon form, and it is also true that:

- Each nonzero row has 1 as its first nonzero element.
- Each column containing a leading 1 has no other nonzero elements.

Definition. A matrix (whether square or rectangular) is in the column echelon form if:

- The first nonzero element of each nonzero column occurs in a row below the first nonzero element in the previous column.
- Columns that are completely zero occur last.

Definition. A matrix is in the *column reduced echelon form* if it is in the column echelon form, and it is also true that:

- Each nonzero column has 1 as its first nonzero element.
- Each row containing a leading 1 has no other nonzero entries.

Note 1. Matrix in the column echelon form or row echelon form has a form of echelon prism or trapezoidal form.

Note 2. The transposed matrix of the row echelon form has column echelon form and *vice versa*.

Theorem (about the rank of matrix in row/column echelon form) The rank of matrix in row(column) echelon form is equal to the number of nonzero rows (columns) of this matrix.

Proof. Suppose the matrix has a row echelon form and the number of nonzero rows is equal to r. To proof this theorem we should find the nonzero minor of the r-th order and to show that all minors of the bigger order are equal to zero.

Since there are only r nonzero rows then each minor of the bigger order if it exists has zero-row and thus it is equal to zero.

Let us consider the following determinant of the r^{th} order with elements from the first r nonzero rows where: the k^{th} column is the column consisting of the elements of the column of the first nonzero element of k^{th} row, k varies from 1 to r. At this choice of columns we get the upper triangular determinant with nonzero elements on the main diagonal, i.e. nonzero determinant of the r^{th} order. It means that rank of the matrix is equal to r.

In the similar way this theorem can be proven for the matrices in the column echelon form. ***Theorem is proven.***

Example. Let us consider the following matrices

$$A=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},\ B=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \\ 7 & 5 & 9 \end{pmatrix},\ C=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 5 & 9 \end{pmatrix},$$

$$D=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 3 & 2 \end{pmatrix}.$$

Matrix A has a row echelon form and matrices B and C have the column echelon form. Matrix D is not in the row echelon form since in other case under the element 1 from the third row it has to be zero element in the fourth row.

The three elementary row operations include:

- interchange of any two rows;
- multiplication of any row by a nonzero number;
- addition of any row multiplied by a nonzero number to another row.

The three elementary column operations include:

- interchange of any two columns;
- multiplication of any column by a nonzero number;
- addition of any column multiplied by a nonzero number to another column.

Definition. The row operations, the column operations and the operation of matrix transposition are called together the elementary matrix manipulations (transformations).

Definition. If matrix B is obtained from matrix A by elementary matrix manipulations then matrices A and B are called equivalent matrices and this relation of equivalence is denoted as A~B.

Note. It is obvious that matrix A can be obtained from B by the set of elementary manipulations which are inverse to initial manipulations applied to A to get B.

Theorem (about ranks of equivalent matrices) If matrices A and B are equivalent matrices then their ranks are equal.

Proof. Since all elementary matrix manipulations can not vanish the basic minor of matrix A according to determinant properties, then this determinant will be nonzero in the matrix B. Let matrix B have the nonzero minor of the bigger order then order of basic minor in A. But it means that this determinant is nonzero in the matrix A, too. We got the contradiction with the definition of basic minor. ***Theorem is proven.***

Corollary. Since the rank of the matrix does not change after elementary matrix manipulations we reduce a matrix to the *row echelon form* or *column echelon form*, because once this form is computed, it is easier to determine the rank.

Example. $A=\begin{pmatrix} 1 & -1 & 0 & 3 & 2 \\ 3 & -1 & 1 & 7 & 5 \\ -1 & 3 & 1 & -5 & -3 \end{pmatrix}\sim$[we add the first row multiplied by (−3) to the second row and then add the same row multiplied by 1 to the third row]~

$$\sim\begin{pmatrix} 1 & -1 & 0 & 3 & 2 \\ 0 & 2 & 1 & -2 & -1 \\ 0 & 2 & 1 & -2 & -1 \end{pmatrix}\sim$$

~[we add to the third row the second one multiplied by (−1)]~

$$\sim\begin{pmatrix} 1 & -1 & 0 & 3 & 2 \\ 0 & 2 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We used only elementary row operations to get matrix in the row echelon form. Since there are two nonzero rows the rank of this matrix is equal to 2.

In this case the minor $\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$, for example, can be chosen as basic minor.

1.15. The Theorem by Kronecker-Kapelly

Let us consider the system of m linear algebraic equations (SLAE) with n unknown variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m \end{cases}$$

If $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, t_i is i-th column of matrix A, then the above system can be rewritten in the following equivalent forms:

$$AX = B \qquad \text{or} \qquad t_1x_1 + t_2x_2 + t_3x_3 + \ldots + t_nx_n = B.$$

Matrix A is called the matrix of the system, B is a column of right sides, X is a column of unknowns.

Definition. If $B \neq 0$ then the system is called inhomogeneous. In other case, i.e. $B = 0$, it is called homogeneous.

Definition. Any set of numbers $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$ is called a solution of the system if after substituting of these numbers in the system one obtains the identity.

Definition. If the system has a solution then it is called a compatible system.

Definition. If the system has no solutions then it is called an incompatible system.

Definition. If the system has the only solution then it is called a definite system.

Definition. If the system has more then one solution then it is called an indefinite system.

Definition. Matrix $A^* = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$ is called an extended matrix of the system.

Note. To differ elements of the matrix *A* from elements of the matrix *B* the extended matrix A^* is usually written down as

$$A^* = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right).$$

Theorem (Theorem by Kronecker-Kapelly) In order to SLAE be compatible it is necessary and sufficient for the ranks of matrices *A* and A^* to be equal, i.e. $rg(A) = rg(A^*)$.

Proof. Necessity: SLAE has a solution $x_1 = \alpha_1, x_2 = \alpha_2, \dots, x_n = \alpha_n$. From the third record of system we have

$$t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \dots + t_n\alpha_n = B,$$

i.e. *B* which is the last column of A^* is linear combination of the other columns of A^*. It means that *B* does not increase the number of linear independent columns of A^* with respect to *A*, so $rg(A) = rg(A^*)$.

Sufficiency: $rg(A) = rg(A^*) = r$. It means that basic minor of *A* can be chosen as basic minor of A^*. But from the theorem about basic rows and columns it means that *B* is a linear combination of the basic columns, i.e. of some columns of *A*:

$$t_{i_1}\alpha_{i_1} + t_{i_2}\alpha_{i_2} + \ldots + t_{i_r}\alpha_{i_r} = B.$$

Let us complete the sum from the left side of expression to full sum of columns by missing columns multiplied by zeros. Then according to the definition of the solution the coefficients of the obtained sum are solution of the system and the system is compatible. ***Theorem is proven.***

Note. It is simple to prove by means of the rule by Cramer that:

- If $rg(A) = rg(A^*) = n$ then system is definite;
- If $rg(A) = rg(A^*) < n$ then system is indefinite.

Let us demonstrate the second statement on the next example.

Example 1. Let us solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4, \\ -2x_1 + 4x_2 - x_3 + 2x_4 = -3. \end{cases}$$

Since the number of unknowns is greater than the number of equations then $rg(A) < n$ and if there are any solutions then the system is indefinite. Let us write down the extended matrix of the system

$$A^* = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ -2 & 4 & -1 & 2 & -3 \end{array}\right).$$

$$\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 4 - 4 = 0,$$

but

$$\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix} = -1 + 6 = 5 \neq 0.$$

Thus $rg(A) = rg(A^*) = 2 < n = 4$ and system is compatible and indefinite.

Let us rewrite the system by leaving to the left only the unknowns corresponding to basic columns:

$$\begin{cases} x_1 + 3x_3 = 4 + 2x_2 + x_4 \\ -2x_1 - x_3 = -3 - 4x_2 - 2x_4 \end{cases}$$

Since the determinant of the obtained system for variables x_1, x_3 is not equal to zero it can be solved by rule by Cramer.

$$x_1 = \frac{\begin{vmatrix} 4+2x_2+x_4 & 3 \\ -3-4x_2-2x_4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}} = \frac{-4-2x_2-x_4-3(-3-4x_2-2x_4)}{5} = 1+2x_2+x_4,$$

$$x_3 = \frac{\begin{vmatrix} 1 & 4+2x_2+x_4 \\ -2 & -3-4x_2-2x_4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}} = \frac{-3-4x_2-2x_4+2(4+2x_2+x_4)}{5} = 1,$$

x_2, x_4 are arbitrary.

Note, that some unknowns are expressed through the others. By assigning any values to x_2, x_4 we get a lot of particular solutions of this system.

Example 2. Let us solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4, \\ -2x_1 + 4x_2 - x_3 + 2x_4 = -3, \\ -x_1 + 2x_2 + 2x_3 + x_4 = 1. \end{cases}$$

We will write down the extended matrix of the system and by means of elementary row operations reduce this matrix to row echelon form. Since we work only with rows what is equivalent to elementary operations (summarizing, adding, subtracting, multiplying by nonzero numbers, changing of the order) on equations, the system stays the same.

$$A^* = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ -2 & 4 & -1 & 2 & -3 \\ -1 & 2 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since $rg(A) = rg(A^*) = 2 < n = 4$ the system is compatible and indefinite.

It is appeared that the third equation in the initial system is linear combination of others equations. So to find solution it is enough to consider only the first two equations what was done in the Example 1.

To check the result we write down the system corresponding to the last extended matrix and compare solutions:

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ 0x_1 + 0x_2 + 1x_3 + 0x_4 = 1 \end{cases} \Leftrightarrow$$

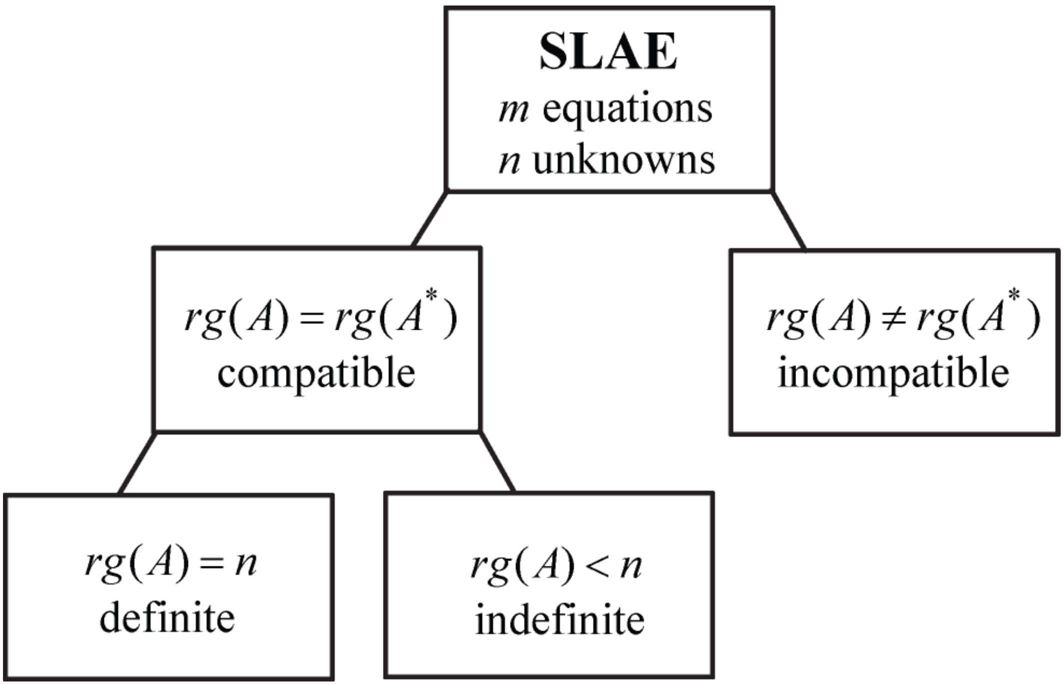
$$\Leftrightarrow \begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ x_3 = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 4 + 2x_2 - 3x_3 + x_4 \\ x_3 = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 1 + 2x_2 + x_4 \\ x_3 = 1 \end{cases} \qquad x_2, x_4 \text{ are arbitrary.}$$

Note 1. The solution of the indefinite system written as function of some arbitrary values is called *the general solution* of the system. Any solution calculated from general by substituting some certain values instead of arbitrary ones is called *the particular solution*.

Note 2. The plan to investigate the SLAE on compatibility can be described by the following diagram:



1.16. Homogeneous Systems
Construction of the Fundamental System of Solutions

Let us consider the homogeneous system of m linear algebraic equations with n unknown variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{cases} \quad \text{or} \quad AX = 0 \quad \text{or} \quad t_1x_1 + t_2x_2 + \ldots + t_nx_n = 0$$

Since $B = 0$ in the homogeneous system (HS) and zero column does not increase the number of linear independent columns in the extended matrix with respect to matrix of the system, *the homogeneous system is always compatible*.

Actually, It is obvious, since the homogeneous system always has a zero (trivial) solution. The question is when does it have nontrivial solution?

Theorem. For the homogeneous system to have nontrivial solution it is necessary and sufficient that $rg(A) < n$.

Proof. Necessity: If we have nontrivial solution then

$$t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \ldots + t_n\alpha_n = 0,\ |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \neq 0.$$

But it means that columns of the matrix A are linear dependent so $rg(A) \neq n$ and thus $rg(A) < n$.

Sufficiency: If $rg(A) < n$ then n columns of the matrix A are linear dependent and there is a set of numbers such that

$$|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \neq 0 \text{ and } t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \ldots + t_n\alpha_n = 0.$$

It means that this set of numbers is a nontrivial solution of the system. ***Theorem is proven.***

Note. It follows from the theorem, that for the homogeneous system of n equations with n variables to have nontrivial solution it is necessary and sufficient that the determinant of the system matrix is equal to zero, i.e. the homogeneous system with square matrix is indefinite if and only if $\det(A) = 0$.

So, if $rg(A) < n$ then the system $AX = 0$ is indefinite and has infinite number of solutions. But how many of them are linearly independent?

Note 1. When we say about the linear dependence of solutions we consider solutions as columns $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and investigate linear dependence of columns.

Note 2. Linear combination of the solutions of homogeneous system is also a solution of this system. Indeed, suppose Y_1, Y_2 are the solutions of the system $AX = 0$, i.e. $AY_1 = 0, AY_2 = 0$. Then

$$A(\alpha Y_1 + \beta Y_2) = A(\alpha Y_1) + A(\beta Y_2) = \alpha AY_1 + \beta AY_2 = \alpha 0 + \beta 0 = 0,$$

i.e. $\alpha Y_1 + \beta Y_2$ is also a solution.

Definition. Fundamental system of solutions (FSS) of the homogeneous system is any maximum set of linearly independent solutions.

Note. It follows from the definition that:

1) Only indefinite homogeneous systems have FSS.
2) Choice of the FSS is not unique.

Theorem (About Fundamental System of Solutions)

(i) If $r = rg(A) < n$ then the homogeneous system has a fundamental system of (n-r) solutions;

(ii) Any solution of the system is a linear combination solutions from FSS.

Proof. Suppose the basic minor stands in the upper left corner of the matrix *A*. Then the first *r* rows are linearly independent and all other rows (equations) are linear combination of the basic rows and, thus, do not contain helpful information to find a solution. So let us consider only the first *r* rows written in the following form:

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{r1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ \vdots \\ a_{r2} \end{pmatrix} x_2 + \ldots + \begin{pmatrix} a_{1r} \\ \vdots \\ a_{rr} \end{pmatrix} x_r = -\begin{pmatrix} a_{1r+1} \\ \vdots \\ a_{rr+1} \end{pmatrix} x_{r+1} - \ldots - \begin{pmatrix} a_{1n} \\ \vdots \\ a_{rn} \end{pmatrix} x_n .$$

The determinant of the obtained system for the unknowns $x_1, x_2, \ldots, x_r$ is not equal to zero, i.e. it is basic minor, and we can find values of the unknowns $x_1, x_2, \ldots, x_r$ as functions of other unknowns by means of rule by Cramer. In this case substituting instead of unknowns $x_{r+1}, x_{r+2}, \ldots, x_n$ some values, we get particular solutions of the initial system. Let us consider the following set of (*n*-*r*) particular solutions:

$$\begin{matrix} x_{r+1} = 1 \\ x_{r+2} = 0 \\ x_{r+3} = 0 \\ \vdots \\ x_n = 0 \end{matrix} \Rightarrow X_1 = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1r} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{matrix} x_{r+1} = 0 \\ x_{r+2} = 1 \\ x_{r+3} = 0 \\ \vdots \\ x_n = 0 \end{matrix} \Rightarrow X_2 = \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2r} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{matrix} x_{r+1} = 0 \\ x_{r+2} = 0 \\ x_{r+3} = 1 \\ \vdots \\ x_n = 0 \end{matrix} \Rightarrow X_3 = \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \\ \vdots \\ \alpha_{3r} \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

$$\ldots, \begin{matrix} x_{r+1} = 0 \\ x_{r+2} = 0 \\ x_{r+3} = 0 \\ \vdots \\ x_n = 1 \end{matrix} \Rightarrow X_{n-r} = \begin{pmatrix} \alpha_{n-r1} \\ \alpha_{n-r2} \\ \vdots \\ \alpha_{n-rr} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The matrix of the order *n* by (*n*-*r*) constructed on these columns has rank equal to (*n*-*r*) since there is unit matrix of the (*n*-*r*)th order in the bottom of it. It means that all these columns (solutions) are linearly independent.

Let us consider now an arbitrary solution of the system $X_0 = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$. Then

$$Y = X_0 - q_{r+1}X_1 - q_{r+2}X_2 - \ldots - q_n X_{n-r} =$$

$$=\begin{pmatrix} q_1 - & q_{r+1}\alpha_{11} & -q_{r+2}\alpha_{21} & -q_{r+3}\alpha_{31} & -\ldots & -q_n\alpha_{n-r1} \\ q_2 - & q_{r+1}\alpha_{12} & -q_{r+2}\alpha_{22} & -q_{r+3}\alpha_{32} & -\ldots & -q_n\alpha_{n-r2} \\ \vdots & & & & & \\ q_r - & q_{r+1}\alpha_{1r} & -q_{r+2}\alpha_{2r} & -q_{r+3}\alpha_{3r} & -\ldots & -q_n\alpha_{n-rr} \\ q_{r+1} - & q_{r+1}\cdot 1 & -q_{r+2}\cdot 0 & -q_{r+3}\cdot 0 & -\ldots & -q_n\cdot 0 \\ q_{r+2} - & q_{r+1}\cdot 0 & -q_{r+2}\cdot 1 & -q_{r+3}\cdot 0 & -\ldots & -q_n\cdot 0 \\ q_{r+3} - & q_{r+1}\cdot 0 & -q_{r+2}\cdot 0 & -q_{r+3}\cdot 1 & -\ldots & -q_n\cdot 0 \\ \vdots & & & & & \\ q_n - & q_{r+1}\cdot 0 & -q_{r+2}\cdot 0 & -q_{r+3}\cdot 0 & -\ldots & -q_n\cdot 1 \end{pmatrix} =$$

$$=\begin{pmatrix}\gamma_1 & \gamma_2 & \ldots & \gamma_r & 0 & \ldots & 0\end{pmatrix}^T$$

is a solution as well, and thus

$$\gamma_1 t_1 + \gamma_2 t_2 + \ldots + \gamma_r t_r + 0t_{r+1} + \ldots + 0t_n = \gamma_1 t_1 + \gamma_2 t_2 + \ldots + \gamma_r t_r = 0\,.$$

Since we obtained zero linear combination of the basic linearly independent columns then $\gamma_1 = 0, \gamma_2 = 0, \ldots, \gamma_r = 0$, i.e.

$$Y = X_0 - q_{r+1}X_1 - q_{r+2}X_2 - \ldots - q_n X_{n-r} = 0$$

and

$$X_0 = q_{r+1}X_1 + q_{r+2}X_2 + \ldots + q_n X_{n-r}\,.$$

It means that any other solution is linear combination of X_1, X_2, $\ldots X_{n-r}$ and can not increase number of linearly independent columns. Thus X_1, X_2, $\ldots X_{n-r}$ form FSS and any solution is a a linear combination of solutions from FSS.

Theorem is proven.

Theorem (about general solution of inhomogeneous system). General solution of the inhomogeneous system $AX = B$ is a sum of the particular solution of the inhomogeneous system and linear combination of solutions from the FSS of homogeneous system $AX = 0$.

Proof. Suppose X is an arbitrary solution and X_0 is some particular solution of the system $AX = B$.

Then $AY = A(X - X_0) = AX - AX_0 = B - B = 0$ and $Y = X - X_0$ is the solution of the homogeneous system and thus equal to the linear combination of solutions of the FSS.

Thus, $X = Y + X_0$ is a sum of the particular solution of the inhomogeneous system and linear combination of the FSS. ***Theorem is proven.***

1.17. Method by Gauss
(Method of Sequential Elimination of the Unknown Variables)
Method by Jordan-Gauss

Method by Gauss is used to solve the system of the linear algebraic equations with arbitrary numbers of equations and unknowns.

It includes sequential elimination of the variables from equations (i.e. vanishing of its coefficient in the equations) according to the following scheme:

Step 1. Form the extended matrix of the system.

Step 2.

- Choose the leading equation and the leading variable (its coefficient in the leading equation has to be nonzero). Put the row of this equation on the first place. Eliminate the leading variable from the other rows below the leading one (i.e. from other equations) by the elementary row operations.
- Then choose new leading equation and new leading variable. Put the row of this equation on the second place and eliminate new leading variable from all other rows below this one.
- Then choose new equation and new variable and so on.
- After such manipulations the obtained matrix with columns rewritten in the order of the chosen leading variables has the row echelon (or trapezoidal) form.

Note. It is preferred to choose the leading variables in the natural order to get exactly row echelon form of the system matrix.

Step 3. Determine the ranks of the system matrix A and the extended matrix A^* and write down the system corresponding to the obtained extended matrix.

Step 4.

- If $rg(A) = rg(A^*) = n$, where n is a number of unknowns, then the system has the only solution which can be calculated from the obtained system.
- If $rg(A) = rg(A^*) < n$ then choose basic minor of the triangular form (for example, consisting of the columns of leading variables). Unknown variables whose coefficients correspond to this minor are called *basic (or main) variables*. All other are called *free (or independent) variables*. Solve the system expressing the main variables through the free ones. Start from the last equation. The obtained equalities are the general solution of the system. Assigning any values to free variables one gets the particular solution of the system.

Example. Let us solve the system of linear equations by the method by Gauss:

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = -1 \\ 2x_1 - 3x_2 - 2x_3 + 3x_4 = 8 \\ x_1 - x_2 - x_3 + x_4 = 3 \\ 3x_1 + x_2 - 3x_3 - 2x_4 = -1 \end{cases}$$

Let us write down the extended matrix of the system and carry out the transformations:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 2 & -3 & -2 & 3 & 8 \\ 1 & -1 & -1 & 1 & 3 \\ 3 & 1 & -3 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -5 & 0 & 5 & 10 \\ 0 & -2 & 0 & 2 & 4 \\ 0 & -2 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & -2 & 0 & 1 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We got the matrix in the row echelon form. $rg(A) = rg(A^*) = 3$. Number of variables is equal to 4. Thus the system is compatible and indefinite. We should choose main and free variables.

$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$. So, it can be chosen as basic minor and variables x_1, x_2, x_4 are main, x_3 is free.

Let us write down the system:

$$\begin{cases} 1\cdot x_1 + 1\cdot x_2 - 1\cdot x_3 - 1\cdot x_4 = -1 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 - 1\cdot x_4 = -2 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 1\cdot x_4 = 2 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 - x_3 - x_4 = -1 \\ x_2 - x_4 = -2 \\ x_4 = 2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = -1 - x_2 + x_3 + x_4 = 1 + x_3 \\ x_2 = -2 + x_4 = 0 \\ x_4 = 2 \end{cases}$$

So, answer is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 1 \\ 0 \\ x_3 \\ 2 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \ x_3 \in R.$$

Notice, that according to the theorem about general solution of inhomogeneous system, the column at x_3 is a solution of the homogeneous system and free column is a particular solution of the inhomogeneous system. Let us check the result:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : \begin{cases} 1 + 0 - 1 - 0 = 0 \\ 2\cdot 1 - 3\cdot 0 - 2\cdot 1 + 3\cdot 0 = 0 \\ 1 - 0 - 1 + 0 = 0 \\ 3\cdot 1 + 0 - 3\cdot 1 - 2\cdot 0 = 0 \end{cases} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} : \begin{cases} 1 + 0 - 0 - 2 = -1 \\ 2\cdot 1 - 3\cdot 0 - 2\cdot 0 + 3\cdot 2 = 8 \\ 1 - 0 - 0 + 2 = 3 \\ 3\cdot 1 + 0 - 3\cdot 0 - 2\cdot 2 = -1 \end{cases}$$

Note. A modification of the method by Gauss where the leading variable is eliminated not only from the below rows (equations) but from all other rows (equations) is called *the method by Jordan-Gauss*. In this method the extended matrix is reduced to *the row reduced echelon form.*

Example. Let us solve the homogeneous system of equation and find its fundamental system of solutions. Since the difference between the matrix of the system and extended matrix is in zero column we will work only with matrix of the system.

$$\begin{cases} x_1 + x_2 - x_3 - x_4 + x_5 = 0 \\ 2x_1 - 3x_2 - x_3 + 3x_4 - x_5 = 0 \\ x_1 - x_2 - x_3 + x_4 = 0 \end{cases} \Rightarrow$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 2 & -3 & -1 & 3 & -1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & -5 & 1 & 5 & -3 \\ 0 & -2 & 0 & 2 & -1 \end{pmatrix} \sim$$

~[Add to the second row the third one multiplied by (–3)] ~

$$\sim \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{pmatrix} \sim$$

~[Add to the first row the third one multiplied by 1] ~

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}.$$

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$. So the leading variables x_1, x_2, x_5 are main and x_3, x_4 are free.

New system is $\begin{cases} x_1 = 0 \\ x_2 + x_3 - x_4 = 0 \\ -2x_3 + x_5 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 + x_4 \\ x_5 = 2x_3 \end{cases}$ and the general solution is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4, \quad x_3, x_4 \in R. \text{ Here } FSS = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

1.18. Examples of Problems for Practices on Linear Algebra

Practice 1 "Matrices. Operations on Matrices"

1. Reproduce the matrix A of the size 2 by 3 with the elements $a_{ij} = 2i + (-1)^j$.

Answer: $A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 5 & 3 \end{pmatrix}$.

2. Find the matrix $A + 3B$ if $A = \begin{pmatrix} 4 & 0 & -1 \\ 1 & 5 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 & 1 \\ -3 & -2 & 1 \end{pmatrix}$.

Answer: $A + 3B = \begin{pmatrix} 10 & 15 & 2 \\ -8 & -1 & 6 \end{pmatrix}$.

3. Find the matrix $2A - 3B$ if $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

Answer: $2A - 3B = \begin{pmatrix} -5 & -3 & -5 \\ 3 & 2 & -5 \\ -4 & -1 & 1 \end{pmatrix}$.

4. Find a matrix X from the equation: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} + 2X = \begin{pmatrix} 1 & 1 & -3 \\ 3 & 2 & 3 \end{pmatrix}$.

Answer: $X = \begin{pmatrix} 0 & 0 & -1.5 \\ 1.5 & 0.5 & 0.5 \end{pmatrix}$.

5. Find a matrix X from the equation: $3\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix} - 5X^T = 2\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$.

Answer: $X = \frac{1}{5}\begin{pmatrix} 12 & 0 \\ -3 & 11 \end{pmatrix}^T = \begin{pmatrix} 2.4 & -0.6 \\ 0 & 2.2 \end{pmatrix}$.

6. Find all possible pairwise products of these matrices (products of two matrices): $A = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ -1 & -3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix}$.

Answer: $AB=\begin{pmatrix}-1 & -2\\ 3 & 12\end{pmatrix}$, $BA=\begin{pmatrix}17 & 4 & 14\\ 22 & 5 & 19\\ -13 & -3 & -11\end{pmatrix}$, $AC=\begin{pmatrix}-1 & 1 & 3\\ 1 & 6 & 3\end{pmatrix}$,

$CB=\begin{pmatrix}4 & 12\\ 3 & 7\\ -3 & -10\end{pmatrix}$, CA and BC do not exist.

7. Find a matrix X such that the product of the matrices X and A gives: a) the first row of A; b) the second column of A; c) the matrix $\begin{pmatrix}a_{11}+3a_{21} & a_{12}+3a_{22} & a_{13}+3a_{23}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$. Here $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$.

Answer: a) $XA=\begin{pmatrix}1 & 0 & 0\end{pmatrix}A$; b) $AX=A\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$; c) $XA=\begin{pmatrix}1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$.

8. Take any matrix of the size 2 by 4 and check that both products $A\cdot A^T$ and $A^T\cdot A$ are symmetrical matrices. Explain the result.

Answer: Due to the property $(AB)^T=B^TA^T$ we get $\left(AA^T\right)^T=\left(A^T\right)^TA^T=AA^T$.

9. Find A^5 if $A=\begin{pmatrix}1 & 2\\ 0 & -2\end{pmatrix}$.

Answer: $A^5=\begin{pmatrix}1 & 22\\ 0 & -32\end{pmatrix}$.

10. Check that $A=\begin{pmatrix}2 & -1\\ -3 & 3\end{pmatrix}$ is a root of matrix equation $x^2-5x+3=0$.

11. Find A^n (n is a natural number) if: a) $A=\begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}$; b) $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$; c) $A=\begin{pmatrix}a & 1\\ 0 & a\end{pmatrix}$; d) $A=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$; e) $A=\begin{pmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{pmatrix}$.

Answer: a) $A^n=\begin{cases}E, \text{ for even } n,\\ A, \text{ for odd } n;\end{cases}$; b) $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$; c) $\begin{pmatrix}a^n & na^{n-1}\\ 0 & a^n\end{pmatrix}$; d) A; e) $\begin{pmatrix}\cos nx & -\sin nx\\ \sin nx & \cos nx\end{pmatrix}$.

12. Find all matrices commutative with matrix: a) $A=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$; b) $B=\begin{pmatrix}4 & 1\\ 0 & 0\end{pmatrix}$.

Answer: a) $\begin{pmatrix}a & b\\ b & a+b\end{pmatrix}$, $a\in R, b\in R$; b) $\begin{pmatrix}a & b\\ 0 & a-4b\end{pmatrix}$, $a\in R, b\in R$.

Tasks for self-studying on topic “Matrices. Operations on Matrices”

1	Calculate $3\begin{pmatrix}2 & 4\\ 1 & -1\\ 0 & 2\end{pmatrix}^T+6\begin{pmatrix}-1 & 1 & 4\\ 2 & 1 & 3\end{pmatrix}$.
2	Determine the size of matrix A if: a) $\begin{pmatrix}1 & 2 & 3\end{pmatrix}^T\cdot A=\begin{pmatrix}4 & 5 & 6\end{pmatrix}^T$; b) $\begin{pmatrix}1 & 2\end{pmatrix}^T\cdot A=\begin{pmatrix}3 & 4 & 5\\ 6 & 7 & 8\end{pmatrix}$; c) $A\cdot\begin{pmatrix}1 & 2 & 3\end{pmatrix}=\begin{pmatrix}4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$.
3	Calculate $\frac{1}{3}\begin{pmatrix}1 & 0 & 2\\ 4 & 1 & 0\\ 0 & 1 & 1\\ 2 & 3 & -2\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}$.
4	Find $f(A)$ if $A=\begin{pmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 1 & 2 & -1\end{pmatrix}$ and $f(x)=3x^2-5x+4$.
5	Find A^n if: a) $A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$; b) $A=\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}$.

Answers: **1.** $\begin{pmatrix}0 & 9 & 24\\ 24 & 3 & 24\end{pmatrix}$; **2.** a) 1 by 1 , b) 1 by 3, c) 2 by 1; **3.** $\begin{pmatrix}2 & 5 & 3 & 7\end{pmatrix}^T$;

4. $\begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 13 & -10 & 12 \end{pmatrix}$; **5.** a) $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A^n = 0, n > 2$ b) $\begin{pmatrix} 0 & 0 \\ (-1)^{n+1} & (-1)^n \end{pmatrix}$.

Practice 2 "Determinants of the Second and the Third Orders"

1. Calculate the determinants:

a) $\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}$; b) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$; c) $\begin{vmatrix} 3 & 2 \\ 8 & 5 \end{vmatrix}$; d) $\begin{vmatrix} 6 & 9 \\ 8 & 12 \end{vmatrix}$;

e) $\begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix}$; f) $\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$; g) $\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix}$; h) $\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix}$;

i) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$; j) $\begin{vmatrix} n+1 & n \\ n & n-1 \end{vmatrix}$; k) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$; l) $\begin{vmatrix} a^2+ab+b^2 & a+b \\ a^2-ab+b^2 & a-b \end{vmatrix}$;

m) $\begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}$; n) $\begin{vmatrix} \sin x+\sin y & \cos x+\cos y \\ \cos x-\cos y & \sin x-\sin y \end{vmatrix}$; o); $\begin{vmatrix} \sin x & \cos x \\ \sin y & \cos y \end{vmatrix}$

p) $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$; q) $\begin{vmatrix} 1 & -\ln b \\ 1 & \ln a \end{vmatrix}$.

Answer: a) 1; b) -2; c) -1; d) 0;
e) -6; f) 3; g) 1; h) 9;
i) 0; j) -1; k) $4ab$; l) $-2b^3$;
m) 1; n) 0; o) $\sin(x-y)$; p) 0; q) $\ln(ab)$.

2. Solve the equations: a) $\begin{vmatrix} \sin x & \sin 5x \\ 1 & 1 \end{vmatrix} = 0$; b) $\begin{vmatrix} 2 & \log_2(5x-4) \\ 1 & \log_2 x \end{vmatrix} = 0$.

Answer: a) $x = \dfrac{\pi k}{2}, k \in Z$ or $x = \dfrac{\pi}{6} + \dfrac{\pi k}{3}, k \in Z$; b) $x_1 = 1, x_2 = 4$.

3. Solve an inequality: $\begin{vmatrix} 4+6x & 1 \\ 9+x^2 & 1 \end{vmatrix} > 0$.

Answer: $x \in (1;5)$.

4. Calculate the determinants by the rule of triangles:

$$\text{a) } \begin{vmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix}; \qquad \text{b) } \begin{vmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 2 \end{vmatrix}; \qquad \text{c) } \begin{vmatrix} 4 & -3 & 5 \\ 3 & -2 & 8 \\ 1 & -7 & -5 \end{vmatrix};$$

$$\text{d) } \begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}; \qquad \text{e) } \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}; \qquad \text{f) } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

Answer: a) 40; b) -3; c) 100; d) -5; e) 1; f) 0.

5. Find all minors and algebraic cofactors of the elements of the first row and the second column of $A = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{pmatrix}$.

Answer: $\begin{cases} M_{11} = 1 \\ M_{12} = 13 \\ M_{13} = 17 \end{cases} \Rightarrow \begin{cases} A_{11} = 1 \\ A_{12} = -13 \\ A_{13} = 17 \end{cases}$ and $\begin{cases} M_{12} = 13 \\ M_{22} = 3 \\ M_{32} = -11 \end{cases} \Rightarrow \begin{cases} A_{12} = -13 \\ A_{22} = 3 \\ A_{32} = 11 \end{cases}$.

6. Calculate the determinant of matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{pmatrix}$ expanding along the the first row and down the second column.

Answer: $\det A = 40$.

7. Calculate the determinants of matrices:

$$\text{a) } \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}; \text{ b) } \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn-1} & a_{nn} \end{pmatrix}; \text{ c) } \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & & \vdots \\ \vdots & \vdots & 0 & \ddots & a_{n-1n} \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}.$$

Answer: a)-c) $\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$.

8. Calculate the algebraic cofactors A_{23} and A_{24} of the matrix $A=\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 3 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$.

Answer: $A_{23}=-3$, $A_{24}=-3$.

9. Calculate the determinants expanding by cofactors:

$$\text{a) } \begin{vmatrix} 3 & 1 & 2 \\ -1 & 2 & 1 \\ 4 & 1 & 0 \end{vmatrix}; \qquad \text{b) } \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}; \qquad \text{c) } \begin{vmatrix} 5 & 6 & 3 \\ 0 & 1 & 0 \\ 7 & 4 & 5 \end{vmatrix}.$$

Answer: a) -17; b) 2; c) 4.

Tasks for self-studying on topic "Determinants of the 2nd and the 3rd orders"

1	Calculate $\begin{vmatrix} \dfrac{(1-t)^2}{1+t^2} & \dfrac{2t}{1+t^2} \\ \dfrac{2t}{1+t^2} & -\dfrac{(1+t)^2}{1+t^2} \end{vmatrix}$.
2	Solve an equation $\begin{vmatrix} \dfrac{x^2-39}{2\sqrt{x^2-24}} & 1 \\ & 1 \end{vmatrix}=0$.
3	Calculate: a) $\begin{vmatrix} 1 & 1 & -1 \\ 4 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix}$; b) $\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix}$; c) $\begin{vmatrix} 4 & 2 & -1 \\ 5 & 3 & -2 \\ 3 & 2 & -1 \end{vmatrix}$.
4	Find all algebraic cofactors of elements of the last row of $A=\begin{pmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{pmatrix}$.
5	Calculate the algebraic cofactor A_{31} of $A=\begin{pmatrix} 1 & 3 & 4 & -5 \\ 4 & 8 & 7 & -2 \\ 7 & 100 & 4 & -3 \\ 20 & 2 & -1 & 8 \end{pmatrix}$.

Answers: **1.** -1; **2.** $x=\pm 7$; **3.** a) -22; b) 0; c) 1; **4.** $A_{31}=-7, A_{32}=11, A_{33}=1$; **5.** 0.

Practice 3 "Determinants of the Higher Orders"

1. Calculate the determinants of the 4th order:

$$\text{a)}\begin{vmatrix}1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{vmatrix};\quad \text{b)}\begin{vmatrix}1&1&1&1\\0&2&1&1\\0&0&3&1\\2&2&2&3\end{vmatrix};\quad \text{c)}\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix};$$

$$\text{d)}\begin{vmatrix}1&1&1&1\\2&-1&4&1\\-3&2&-1&5\\4&3&1&-1\end{vmatrix};\quad \text{e)}\begin{vmatrix}4&3&2&10\\-4&2&5&5\\2&4&3&10\\0&4&-1&2\end{vmatrix};\quad \text{f)}\begin{vmatrix}7&6&3&7\\3&5&7&2\\5&4&3&5\\5&6&5&4\end{vmatrix}.$$

Answer: a) -8; b) 6; c) -3; d) 5; e) -30; f) -10.

2. Calculate the determinants of the 5th order:

$$\text{a)}\begin{vmatrix}0&1&2&0&0\\0&3&4&0&0\\1&-1&2&-2&3\\2&4&5&-1&4\\3&-1&3&-2&5\end{vmatrix};\quad \text{b)}\begin{vmatrix}0&3&1&0&3\\1&4&-2&3&4\\0&5&1&0&-1\\2&-1&2&-1&2\\0&2&1&0&1\end{vmatrix};\quad \text{c)}\begin{vmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{vmatrix};$$

$$\text{d)}\begin{vmatrix}4&4&-2&4&2\\6&3&3&-3&3\\-2&-1&0&1&-1\\-2&4&1&-1&5\\2&1&2&3&0\end{vmatrix};\quad \text{e)}\begin{vmatrix}1&1&1&1&1\\1&2&1&1&1\\1&1&3&1&1\\1&1&1&4&1\\1&1&1&1&5\end{vmatrix};\quad \text{f)}\begin{vmatrix}1&5&5&5&5\\5&2&5&5&5\\5&5&3&5&5\\5&5&5&4&5\\5&5&5&5&5\end{vmatrix}.$$

Answer: a) 8; b) 56; c) 4; d) 84; e) 24; f) 120.

3. Calculate the determinants of the nth order:

$$\text{a)}\begin{vmatrix}1&n&n&\dots&n\\n&2&n&n&\vdots\\n&n&3&n&n\\\vdots&n&n&\ddots&n\\n&\dots&n&n&n\end{vmatrix};\ \text{b)}\begin{vmatrix}x&-y&0&\dots&\dots&0\\0&x&-y&0&&\vdots\\\vdots&0&x&-y&0&\vdots\\\vdots&&&\ddots&&0\\0&0&\dots&0&x&-y\\-y&0&\dots&&0&x\end{vmatrix};\ \text{c)}\begin{vmatrix}1&x_1&x_1^2&\dots&x_1^{n-1}\\1&x_2&x_2^2&\dots&x_2^{n-1}\\1&x_3&x_3^2&\dots&x_3^{n-1}\\\vdots&&&&\\1&x_n&x_n^2&\dots&x_n^{n-1}\end{vmatrix}.$$

Answer: a) $n!$; b) $x^n - y^n$; c) $\prod\limits_{\substack{i,j=1 \\ (j<j)}}^{n} (x_i - x_j)$.

Tasks for self-studying on topic "Determinants of the Higher Orders"

1	Calculate $\begin{vmatrix} -z & 1 & 0 & -1 \\ -1 & -z & 0 & 1 \\ 1 & -1 & 0 & -z \\ 1 & 2 & 3 & 4 \end{vmatrix}$.
2	Calculate: a) $\begin{vmatrix} -3 & 9 & 3 & 6 \\ -5 & 8 & 2 & 7 \\ 4 & -5 & -3 & -2 \\ 7 & -8 & -4 & -5 \end{vmatrix}$; b) $\begin{vmatrix} 3 & 1 & 6 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 2 & 3 \end{vmatrix}$.
3	Calculate the determinant of A^3 if $A = \frac{1}{2}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.
4	Calculate $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 0 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix}$.
5	Calculate $\begin{vmatrix} n & 1 & 1 & \dots & 1 \\ 1 & n-1 & 1 & \dots & 1 \\ 1 & 1 & n-2 & & 1 \\ \vdots & & & \ddots & 1 \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix}$.

Answers: **1.** $3z^3 + 9z$; **2.** a) 18, b) 0; **3.** -1; **4.** -8; **5.** $(n-1)!$.

Practice 4 "Rule by Cramer. Inverse Matrix"

1. Find the solutions of the following systems:

$$\text{a) } \begin{cases} x_1 + 2x_2 = 1, \\ 3x_1 + 5x_2 = 2. \end{cases} \quad \text{b) } \begin{cases} 12x_1 + x_2 = 3, \\ 2x_1 - x_2 = -3. \end{cases}$$

Answer: a) $x_1 = -1, x_2 = 1$; b) $x_1 = 0, x_2 = 3$.

2. Solve the system of linear algebraic equations:

$$\begin{cases} x_1 + x_2 + x_3 = 2, \\ 2x_1 - x_2 + x_3 = 1, \\ x_1 - x_2 = 0. \end{cases}$$

Answer: $x_1 = 1, x_2 = 1, x_3 = 0$.

3. Find the value of parameter λ such that $x_4 = -1$:

$$\begin{cases} 2x_1 + x_2 + x_3 = 4, \\ 4x_2 - x_3 + x_4 = \lambda, \\ x_1 - x_2 + 2x_3 + 2x_4 = 0, \\ 3x_1 - 2x_3 + 3x_4 = -2. \end{cases}$$

Answer: $\lambda = 2$.

4. Find the matrices inverse to the given matrices:

$$\text{a) } A = \begin{pmatrix} 5 & 3 \\ -4 & 1 \end{pmatrix} \quad \text{b) } B = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Answer: a) $A^{-1} = \dfrac{1}{17}\begin{pmatrix} 1 & -3 \\ 4 & 5 \end{pmatrix}$; b) $B^{-1} = \dfrac{1}{11}\begin{pmatrix} -1 & 3 & 1 \\ 1 & -3 & 10 \\ 4 & -1 & -4 \end{pmatrix}$.

5. Solve the matrix equations:

$$\text{a) } X\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}; \text{ b) } \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} X \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} = 56\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}.$$

Answer: a) $X = \begin{pmatrix} -\dfrac{11}{2} & \dfrac{5}{2} \\ -9 & 4 \end{pmatrix}$; b) $X = \begin{pmatrix} -4 & -32 \\ -1 & 20 \end{pmatrix}$.

6. Solve a matrix equation:

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} X = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

Answer: $X = \begin{pmatrix} -10 & 5 & 1 \\ -21 & 10 & 0 \\ -18 & 8 & 1 \end{pmatrix}.$

7. Solve the system of equations by means of the inverse matrix:

$$\begin{cases} x_1 + 3x_2 + 2x_3 = 5, \\ 4x_1 + x_3 = 3, \\ -x_1 + 2x_2 + x_3 = 2. \end{cases}.$$

Answer: $x_1 = 1, x_2 = 2, x_3 = -1$.

Tasks for self-studying on topic "Rule by Cramer. Inverse Matrix"

1	Find the solution of the system $\begin{cases} 2x_1 - 2x_2 + 3x_3 + 1 = 0, \\ x_1 - x_2 + 2x_3 + 1 = 0, \\ -2x_1 + x_2 - 2x_3 = 0. \end{cases}.$
2	Solve a matrix equation $X\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$
3	Find X if $A(B + A) = -E^7$, where $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$
4	Solve the system of equations by means of the inverse matrix: $\begin{cases} 3x_1 + x_2 + x_3 = 1, \\ x_1 + 3x_2 + x_3 = 1, \\ x_1 + x_2 + 3x_3 = 1. \end{cases}.$
5	Find the determinant of matrix X if $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} X^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$

Answers: **1.** $\Delta=1, x_1=1, x_2=0, x_3=-1$; **2.** $X=-\frac{1}{3}\begin{pmatrix} 9 & -1 \\ -6 & 8 \end{pmatrix}$; **3.** $X=\frac{1}{6}\begin{pmatrix} -2 & -2 & 2 \\ 2 & 2 & 1 \\ 4 & -2 & 2 \end{pmatrix}$;

4. $x_1=x_2=x_3=\frac{1}{5}$; **5.** $\det X=-\frac{1}{2}$.

Practice 5 "Rank of the Matrix. Method by Gauss"

1. Determine the ranks of the given matrices:

a) $\begin{pmatrix} 3 & 2 & 1 & 3 \\ -2 & 0 & 4 & 5 \\ 1 & 0 & 3 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -1 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & -2 & 3 & -2 \\ 3 & 0 & 1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 1 & 7 & 1 \\ 0 & 2 & -4 & -4 \\ 2 & 3 & 0 & -4 \\ 0 & 5 & 10 & -10 \\ -1 & -4 & 5 & 7 \end{pmatrix}$.

Answer: a) 3; b) 2; c) 3.

2. Investigate these system on compatibility and find the solutions by method by Gauss (for compatible systems):

a) $\begin{cases} x_1+2x_2-x_3=3, \\ 2x_1-x_2+x_3=1, \\ x_1+5x_2+3x_3=6, \\ 2x_1+2x_2+5x_3=4. \end{cases}$ b) $\begin{cases} x_1+3x_2+2x_3=5, \\ 4x_1+x_3=3, \\ -x_1+2x_2+x_3=2, \\ 3x_1+2x_2+2x_3=4. \end{cases}$ c) $\begin{cases} 2x_1+x_2+x_3-2x_4+2x_5=4, \\ x_1+3x_2+4x_3-3x_4+x_5=6, \\ 3x_1-x_2-2x_3-x_4+3x_5=3, \\ x_1-2x_2-3x_3+x_4+x_5=-2. \end{cases}$

d) $\begin{cases} x_1+2x_2+3x_3-x_4=0, \\ x_1-x_2+x_3+2x_4=4, \\ x_1+5x_2+5x_3-4x_4=-4, \\ x_1+8x_2+7x_3-7x_4=-8. \end{cases}$ e) $\begin{cases} 2x_1-2x_2+x_3-x_4=0, \\ x_1-x_2+x_4=1, \\ 2x_1-3x_2+2x_3-x_4=0, \\ x_1-2x_2+x_3+x_4=1. \end{cases}$

f) $\begin{cases} 2x_1-x_2+3x_3+2x_4+x_5=1, \\ 3x_1+5x_2+5x_3-4x_4+3x_5=8, \\ x_1-7x_2+x_3+8x_4-x_5=-6. \end{cases}$ g) $\begin{cases} x_1-x_2+2x_3-x_4=0, \\ 2x_1-2x_2+x_3+2x_4=0, \\ x_1-x_2+2x_3=0, \\ 3x_1-2x_3-2x_4=0. \end{cases}$

Answer: a) $x_1=1, x_2=1, x_3=0$; b) no solutions; c) no solutions;

d) $x_1 = \frac{5}{2}x_2 - \frac{7}{2}x_4 + 6,\ x_3 = -\frac{3}{2}x_2 + \frac{3}{2}x_4 - 2\ (x_2 \in R, x_4 \in R)$;

e) $x_1 = 2x_4 - 1,\ x_2 = x_3 = 3x_4 - 2\ (x_4 \in R)$;

f) $x_1 = -\frac{20}{13}x_3 - \frac{6}{13}x_4 - \frac{8}{13}x_5 + 1,\ x_2 = -\frac{1}{13}x_3 + \frac{14}{13}x_4 - \frac{3}{13}x_5 + 1\ \ (x_3 \in R, x_4 \in R, x_5 \in R)$;

g) $x_1 = x_2 = x_3 = x_4 = 0$.

3. Find FSS of homogeneous systems:

a) $\begin{cases} x_1 - x_2 + 2x_3 - x_4 = 0, \\ 2x_1 - 2x_2 + x_3 + 2x_4 = 0, \\ 3x_1 - 3x_2 - x_3 = 0, \\ x_1 - x_2 - x_3 + 3x_4 = 0. \end{cases}$ b) $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0, \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + x_5 = 0, \\ 3x_1 + 4x_2 + 5x_3 + x_4 + 2x_5 = 0, \\ x_1 + 3x_2 + 5x_3 + 12x_4 + 9x_5 = 0, \\ 4x_1 + 5x_2 + 6x_3 - 3x_4 + 3x_5 = 0. \end{cases}$

Answer: a) $FSS = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$; b). $FSS = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 15 \\ -12 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

4. Investigate the system and find solution for different values of parameter λ:

$$\begin{cases} 5x_1 - 3x_2 + 2x_3 + 4x_4 = 3, \\ 4x_1 - 2x_2 + 3x_3 + 7x_4 = 1, \\ 8x_1 - 6x_2 - x_3 - 5x_4 = 9, \\ 7x_1 - 3x_2 + 7x_3 + 17x_4 = \lambda. \end{cases}$$

Answer: For $\lambda \neq 0$ the system is incompatible, for $\lambda = 0$ it is compatible and indefinite with solutions

$$x_1 = \frac{-5x_3 - 13x_4 - 3}{2},\ x_2 = \frac{-7x_3 - 19x_4 - 7}{2}\ (x_3 \in R, x_4 \in R).$$

5. Find the value of λ for the system to be indefinite:

$$\begin{cases} \lambda x_1 - x_2 + 2x_3 - x_4 = 0, \\ x_1 + 2x_2 + x_3 + 2x_4 = 0, \\ x_1 - x_2 + x_3 = 0, \\ 3x_1 + x_3 + x_4 = 0. \end{cases}$$

Answer: $\lambda = -8$.

Tasks for self-studying on topic "Rank of the Matrix. Method by Gauss"

1	Investigate these systems on compatibility and find the solutions by method by Gauss (for compatible systems): a) $\begin{cases} x_1+2x_2-3x_3=0, \\ 3x_1+2x_2-4x_3=2, \\ 2x_1-x_2=2. \end{cases}$ b) $\begin{cases} x_1+x_2-x_3-x_4=0, \\ 2x_1-3x_2-2x_3+3x_4=5, \\ x_1-x_2-x_3+x_4=2, \\ 3x_1+x_2-3x_3-2x_4=1. \end{cases}$
2	Find FSS of homogeneous system $\begin{cases} x_1+2x_2+3x_3+4x_4+5x_5=0, \\ x_1+4x_2+10x_3+x_4+2x_5=0, \\ 3x_1+8x_2+16x_3+9x_4+12x_5=0, \\ x_1-4x_3+7x_4+8x_5=0, \\ 2x_2+7x_3-3x_4-3x_5=0. \end{cases}$
3	Investigate the system and find the solution for different values of parameter λ: $\begin{cases} \lambda x_1+x_2+x_3=1 \\ x_1+\lambda x_2+x_3=1. \\ x_1+x_2+\lambda x_3=1 \end{cases}$

Answers: **1.** a) $x_1=2, x_2=2, x_3=2$; b) $x_1=x_3+1,\ x_2=0,\ x_4=1\ (x_3\in R)$;**2.**

$$FSS=\left\{\begin{pmatrix}4\\-\frac{7}{2}\\1\\0\\0\end{pmatrix},\begin{pmatrix}-7\\\frac{3}{2}\\0\\1\\0\end{pmatrix},\begin{pmatrix}-8\\\frac{3}{2}\\0\\0\\1\end{pmatrix}\right\};$$ **3.** For $\lambda=1$ the system is indefinite with solutions $x_1=1-x_2-x_3\ (x_2\in R, x_3\in R)$, for $\lambda=-2$ the system is incompatible, for other values of parameter the system is definite with solution $x_1=x_2=x_3=\dfrac{1}{\lambda+2}$.

1.19. Individual Tasks to Chapter 1

Task 1. Solve the following matrix equation:

1.1. $4\begin{pmatrix} 3 & 1 & -1 \\ 2 & 5 & 0 \end{pmatrix} + 2X^T = 3\begin{pmatrix} 0 & -2 & 3 \\ 1 & 1 & 4 \end{pmatrix}$;

1.2. $2\begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 0 \end{pmatrix} + 5X^T = 3\begin{pmatrix} 1 & 2 & 0 \\ 5 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$;

1.3. $2\begin{pmatrix} 3 & 1 & -1 \\ 2 & 5 & 0 \end{pmatrix}^T - 4X = 3\begin{pmatrix} 0 & -2 & 3 \\ 1 & 1 & 4 \end{pmatrix}^T$;

1.4. $4\begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 5 & 0 \end{pmatrix} - 5X^T = 3\begin{pmatrix} 0 & 7 \\ -1 & 1 \\ -2 & 3 \end{pmatrix}$;

1.5. $2\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}^T + 4X = 5\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$;

1.6. $4\begin{pmatrix} 3 & 1 & -1 \\ 2 & 5 & 0 \end{pmatrix}^T - 2X = 3\begin{pmatrix} 0 & -2 & 3 \\ 1 & 1 & 4 \end{pmatrix}^T$;

1.7. $3\begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 0 \end{pmatrix} - 2X^T = 4\begin{pmatrix} 1 & 2 & 0 \\ 5 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$;

1.8. $2\begin{pmatrix} 3 & 4 & -5 \\ 2 & 2 & 0 \end{pmatrix} - 4X^T = 3\begin{pmatrix} 0 & -7 & 3 \\ 4 & 6 & 4 \end{pmatrix}$;

1.9. $-3\begin{pmatrix} 6 & 4 \\ 2 & 3 \\ 5 & 0 \end{pmatrix} + 5X^T = 2\begin{pmatrix} 1 & 7 \\ -1 & 2 \\ -2 & 3 \end{pmatrix}$;

1.10. $5\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}^T + 2X = 5\begin{pmatrix} 4 & 3 \\ -3 & -1 \end{pmatrix}$;

1.11. $2\begin{pmatrix} 3 & -1 & -1 \\ 2 & 6 & 3 \end{pmatrix} + 10X^T = -3\begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 0 \end{pmatrix};$

1.12. $\begin{pmatrix} -1 & 0 & 1 \\ 3 & 2 & 6 \\ 7 & 4 & 0 \end{pmatrix} + 2X^T = 3\begin{pmatrix} 1 & -2 & 0 \\ 5 & 3 & 2 \\ 2 & 5 & 1 \end{pmatrix};$

1.13. $2\begin{pmatrix} 1 & 4 & -1 \\ 2 & 6 & 0 \end{pmatrix}^T - 4X = 3\begin{pmatrix} 0 & -2 & 3 \\ 1 & -1 & 4 \end{pmatrix}^T;$

1.14. $4\begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 6 & 0 \end{pmatrix} + 5X^T = 3\begin{pmatrix} 2 & 7 \\ -1 & 4 \\ -2 & 3 \end{pmatrix};$

1.15. $2\begin{pmatrix} 3 & -2 \\ 4 & 5 \end{pmatrix} - 20X = 5\begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix}^T;$

1.16. $4\begin{pmatrix} 6 & 1 & -1 \\ 1 & 5 & 0 \end{pmatrix} - 5X^T = 3\begin{pmatrix} 1 & -5 & 3 \\ 4 & 1 & 4 \end{pmatrix};$

1.17. $2\begin{pmatrix} -1 & 0 & 1 \\ 2 & -3 & 2 \\ 1 & 4 & 0 \end{pmatrix}^T + 5X = 3\begin{pmatrix} 2 & 2 & 0 \\ 5 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix};$

1.18. $4\begin{pmatrix} 3 & 2 & -1 \\ 1 & 5 & 0 \end{pmatrix}^T - 8X = 5\begin{pmatrix} 0 & -1 & 3 \\ 6 & 1 & 4 \end{pmatrix}^T;$

1.19. $3\begin{pmatrix} 3 & 4 \\ 2 & 0 \\ 5 & 1 \end{pmatrix} - 5X^T = 2\begin{pmatrix} 5 & 7 \\ -1 & 6 \\ -2 & 3 \end{pmatrix};$

1.20. $3\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} - 2X = 5\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}^T;$

1.21. $3\begin{pmatrix} 0 & 1 & -3 \\ 2 & 5 & 0 \end{pmatrix} - 2X^T = 4\begin{pmatrix} 4 & -2 & 3 \\ 1 & -5 & 0 \end{pmatrix};$

1.22. $2\begin{pmatrix} -1 & 0 & 4 \\ 4 & 3 & 2 \\ 1 & 4 & 0 \end{pmatrix} + 5X = 3\begin{pmatrix} 3 & 2 & 0 \\ 5 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}^T$;

1.23. $5\begin{pmatrix} 3 & -1 & 1 \\ 2 & -5 & 0 \end{pmatrix}^T - 4X = 2\begin{pmatrix} 0 & -2 & 3 \\ -1 & 3 & 4 \end{pmatrix}^T$;

1.24. $3\begin{pmatrix} 2 & -2 \\ 2 & 4 \\ 0 & 6 \end{pmatrix} + 20X^T = 2\begin{pmatrix} 0 & 7 \\ -1 & 1 \\ -2 & 3 \end{pmatrix}$;

1.25. $2\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix} - 3X^T = 5\begin{pmatrix} 6 & 3 \\ 3 & -2 \end{pmatrix}$;

1.26. $2\begin{pmatrix} 0 & 1 & -1 \\ 6 & 5 & 3 \end{pmatrix} + 10X^T = 5\begin{pmatrix} 4 & -2 & 3 \\ 1 & 7 & 4 \end{pmatrix}$;

1.27. $2\begin{pmatrix} -1 & 0 & 1 \\ 5 & 3 & 2 \\ 1 & 0 & 3 \end{pmatrix} + 5X^T = 3\begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & -2 \\ -2 & 1 & 1 \end{pmatrix}$;

1.28. $2\begin{pmatrix} 3 & 1 & -1 \\ 2 & 5 & 3 \end{pmatrix} - 4X^T = 3\begin{pmatrix} 5 & -2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$;

1.29. $4\begin{pmatrix} 5 & 4 \\ 2 & 3 \\ 1 & 0 \end{pmatrix} - 5X^T = 3\begin{pmatrix} 0 & 7 \\ -4 & 1 \\ -7 & 3 \end{pmatrix}$;

1.30. $2\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} - 5X = 3\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}^T$.

Task 2. Find the products of matrices *A* and *B*, i.e. $A \cdot B$ and $B \cdot A$ (if they exist):

2.1. $A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 5 \end{pmatrix}$;

2.2. $A=\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix}$;

2.3. $A=\begin{pmatrix} 0 & -1 & 5 \\ 2 & 1 & 1 \\ 4 & 3 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$;

2.4. $A=\begin{pmatrix} 2 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 2 & 3 & -1 \\ -1 & 5 & 0 \end{pmatrix}$;

2.5. $A=\begin{pmatrix} -1 & 3 & 3 \\ -2 & 0 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$;

2.6. $A=\begin{pmatrix} 5 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$;

2.7. $A=\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$, $B=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$;

2.8. $A=\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $B=\begin{pmatrix} 2 & 3 & -1 \\ -1 & 5 & 0 \end{pmatrix}$;

2.9. $A=\begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 0 & 2 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$;

2.10. $A=\begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 0 & 2 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 1 \end{pmatrix}$;

2.11. $A=\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 1 \\ 4 & 3 \\ 0 & 5 \end{pmatrix}$;

2.12. $A=\begin{pmatrix} 0 & 3 \\ -4 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 6 & 0 & 3 \\ -1 & 7 & 2 \end{pmatrix}$;

2.13. $A=\begin{pmatrix} 0 & -1 & 7 \\ 1 & 2 & 0 \\ 4 & 5 & 2 \end{pmatrix},\ B=\begin{pmatrix} 4 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix};$

2.14. $A=\begin{pmatrix} 1 & 2 \end{pmatrix},\ B=\begin{pmatrix} -3 & 2 & -1 \\ 1 & 5 & 4 \end{pmatrix};$

2.15. $A=\begin{pmatrix} -4 & 3 & 3 \\ 0 & 5 & 1 \end{pmatrix},\ B=\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix};$

2.16. $A=\begin{pmatrix} 1 & 4 & 3 \\ 1 & 2 & 5 \end{pmatrix},\ B=\begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix};$

2.17. $A=\begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix},\ B=\begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix};$

2.18. $A=\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix},\ B=\begin{pmatrix} 0 & 2 & 4 \\ -1 & 5 & 0 \end{pmatrix};$

2.19. $A=\begin{pmatrix} 1 & 1 & -3 & 0 \\ 5 & 4 & 2 & 1 \end{pmatrix},\ B=\begin{pmatrix} 1 & 7 \\ -4 & 3 \end{pmatrix};$

2.20. $A=\begin{pmatrix} 1 & 3 & -1 & 3 \\ 1 & 5 & 3 & 1 \end{pmatrix},\ B=\begin{pmatrix} 2 & 1 \end{pmatrix};$

2.21. $A=\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix},\ B=\begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 1 & 5 \end{pmatrix};$

2.22. $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix},\ B=\begin{pmatrix} 1 & 5 & 3 \\ -2 & 1 & 4 \end{pmatrix};$

2.23. $A=\begin{pmatrix} 3 & -1 & 5 \\ 2 & 3 & 0 \\ 4 & 3 & 2 \end{pmatrix},\ B=\begin{pmatrix} 2 & 5 & -1 \\ -1 & 5 & 0 \end{pmatrix};$

2.24. $A=\begin{pmatrix}3 & 1\end{pmatrix},\ B=\begin{pmatrix}3 & 2 & -5\\ -1 & 2 & 0\end{pmatrix};$

2.25. $A=\begin{pmatrix}-7 & 4 & 5\\ -3 & 0 & 2\end{pmatrix},\ B=\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix};$

2.26. $A=\begin{pmatrix}1 & 8 & 2\\ 9 & -2 & 2\end{pmatrix},\ B=\begin{pmatrix}1\\ 2\\ 1\end{pmatrix};$

2.27. $A=\begin{pmatrix}1 & -2\\ -3 & 5\end{pmatrix},\ B=\begin{pmatrix}-2 & 5\\ 1 & -3\end{pmatrix};$

2.28. $A=\begin{pmatrix}1\\ 4\\ -3\end{pmatrix},\ B=\begin{pmatrix}1 & 3 & -1\\ -1 & 5 & 2\end{pmatrix};$

2.29. $A=\begin{pmatrix}0 & 2 & -1 & 0\\ 3 & 2 & -2 & 1\end{pmatrix},\ B=\begin{pmatrix}8 & 2\\ 1 & 5\end{pmatrix};$

2.30. $A=\begin{pmatrix}11 & 2 & -5 & 7\\ 3 & 10 & 4 & 1\end{pmatrix},\ B=\begin{pmatrix}3 & -1\end{pmatrix}.$

Task 3. Find the determinant of the fourth order:

3.1. $\begin{vmatrix}1 & 3 & 1 & 2\\ 3 & 4 & 3 & 2\\ 0 & 2 & 1 & 4\\ 2 & 4 & 2 & 3\end{vmatrix};$ **3.2.** $\begin{vmatrix}-1 & 2 & 4 & -2\\ 3 & 5 & 0 & -1\\ 4 & 3 & 2 & -3\\ 3 & 0 & 4 & 1\end{vmatrix};$ **3.3.** $\begin{vmatrix}2 & 2 & 3 & 4\\ -2 & 1 & -1 & 3\\ 3 & 0 & -1 & 2\\ 4 & 3 & -2 & -1\end{vmatrix};$

3.4. $\begin{vmatrix}1 & 3 & 5 & 7\\ 3 & 0 & 4 & 1\\ 5 & 4 & 1 & 3\\ 7 & 1 & 3 & 5\end{vmatrix};$ **3.5.** $\begin{vmatrix}4 & 3 & 3 & 1\\ 3 & 0 & 3 & 2\\ 3 & 2 & 1 & 4\\ 2 & 4 & 2 & 3\end{vmatrix};$ **3.6.** $\begin{vmatrix}2 & 4 & 2 & 4\\ 4 & 3 & 3 & 7\\ 3 & 1 & 7 & 4\\ 2 & 3 & 4 & 5\end{vmatrix};$

3.7. $\begin{vmatrix} 1 & -3 & 1 & 4 \\ 3 & 5 & 2 & 4 \\ 1 & 2 & 5 & 4 \\ 3 & 2 & 4 & 2 \end{vmatrix}$; **3.8.** $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$; **3.9.** $\begin{vmatrix} 2 & -1 & 1 & -1 \\ 3 & -1 & 1 & 1 \\ 4 & 2 & 1 & -1 \\ 1 & 0 & 1 & 2 \end{vmatrix}$;

3.10. $\begin{vmatrix} 5 & 4 & 3 & 5 \\ 3 & 0 & 5 & 4 \\ 2 & 4 & 2 & -3 \\ 4 & 3 & 4 & -2 \end{vmatrix}$; **3.11.** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 10 \\ 1 & 4 & 10 & 0 \end{vmatrix}$; **3.12.** $\begin{vmatrix} 4 & 5 & 2 & 1 \\ 1 & 4 & 3 & 3 \\ 6 & 3 & 1 & 2 \\ 2 & 2 & 3 & 4 \end{vmatrix}$;

3.13. $\begin{vmatrix} -1 & 3 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \end{vmatrix}$; **3.14.** $\begin{vmatrix} 1 & 1 & 3 & 4 \\ 2 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 4 & 5 & 2 & 3 \end{vmatrix}$; **3.15.** $\begin{vmatrix} 0 & 3 & 2 & 2 \\ 3 & 1 & -2 & 1 \\ 2 & -1 & 1 & 4 \\ 1 & 2 & -3 & 1 \end{vmatrix}$;

3.16. $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 3 & 2 & 1 & -1 \\ 1 & 2 & -3 & 3 \\ 4 & 4 & -2 & 2 \end{vmatrix}$; **3.17.** $\begin{vmatrix} 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix}$; **3.18.** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 5 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$;

3.19. $\begin{vmatrix} 2 & 4 & 0 & 3 \\ 2 & 2 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 4 & 3 & 3 & 1 \end{vmatrix}$; **3.20.** $\begin{vmatrix} -1 & 1 & 3 & 4 \\ 1 & 2 & 1 & 4 \\ -3 & 1 & 2 & -2 \\ 0 & 1 & -2 & 6 \end{vmatrix}$; **3.21.** $\begin{vmatrix} 2 & 1 & 5 & 4 \\ 3 & 6 & 3 & 2 \\ 2 & 5 & 1 & 4 \\ -1 & 3 & 2 & 1 \end{vmatrix}$;

3.22. $\begin{vmatrix} 1 & 1 & 3 & 2 \\ 3 & 2 & 1 & 0 \\ -4 & 0 & 1 & 3 \\ -5 & 2 & 1 & 3 \end{vmatrix}$; **3.23.** $\begin{vmatrix} 3 & 2 & 3 & 4 \\ -1 & 5 & 3 & -8 \\ -1 & 1 & 4 & -5 \\ 2 & 3 & 5 & 15 \end{vmatrix}$; **3.24.** $\begin{vmatrix} 1 & 7 & 4 & 5 \\ 3 & 2 & 1 & 2 \\ 4 & 2 & 4 & -2 \\ -1 & 2 & 3 & 5 \end{vmatrix}$;

3.25. $\begin{vmatrix} 1 & -2 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ 1 & 3 & 1 & 5 \\ 1 & 7 & -4 & 3 \end{vmatrix}$; **3.26.** $\begin{vmatrix} 1 & 4 & 1 & 3 \\ 2 & -1 & 4 & 0 \\ 3 & 2 & 5 & 1 \\ 4 & 5 & 2 & 3 \end{vmatrix}$; **3.27.** $\begin{vmatrix} 4 & 5 & 6 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 4 \end{vmatrix}$;

3.28. $\begin{vmatrix} 1 & 2 & 3 & -2 \\ 3 & 6 & 5 & -4 \\ 1 & 0 & 7 & -4 \\ 2 & 4 & 2 & -3 \end{vmatrix}$; **3.29.** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ -3 & 0 & 3 & 4 \\ 1 & -2 & 4 & 3 \\ -1 & -2 & -3 & 1 \end{vmatrix}$; **3.30.** $\begin{vmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & -1 \\ 2 & -4 & 1 & 3 \\ 3 & -1 & 2 & 5 \end{vmatrix}$.

Task 4. Determine the number of linearly independent rows of the matrix (it is equal to the rank of this matrix):

4.1. $\begin{pmatrix} 2 & 4 & 1 & 0 & -2 \\ 4 & 6 & 3 & 2 & 5 \\ 2 & 6 & 0 & -2 & -11 \\ 10 & 16 & 7 & 4 & 8 \end{pmatrix}$; **4.2.** $\begin{pmatrix} 5 & 1 & -2 & -4 & 5 \\ 2 & 0 & 0 & 3 & -1 \\ 13 & 3 & -6 & -15 & 16 \\ 9 & 1 & -2 & 2 & 3 \end{pmatrix}$;

4.3. $\begin{pmatrix} 1 & -2 & -1 & 3 & 3 \\ 2 & 3 & 3 & 5 & -1 \\ 1 & -9 & -6 & 4 & 10 \\ 5 & 4 & 4 & 13 & 1 \end{pmatrix}$; **4.4.** $\begin{pmatrix} 1 & 2 & -2 & -1 & 3 \\ 5 & 1 & 6 & -1 & 6 \\ -2 & 5 & -12 & -2 & 3 \\ 11 & 4 & 10 & -3 & 15 \end{pmatrix}$;

4.5. $\begin{pmatrix} 1 & 5 & 1 & -1 & 5 \\ 1 & -4 & 5 & -4 & 3 \\ 2 & 19 & -2 & 1 & 12 \\ 3 & -3 & 11 & -9 & 11 \end{pmatrix}$; **4.6.** $\begin{pmatrix} 2 & 2 & 4 & -3 & 3 \\ 2 & 0 & 2 & -1 & 6 \\ 4 & 6 & 10 & -8 & 3 \\ 6 & 2 & 8 & -5 & 15 \end{pmatrix}$;

4.7. $\begin{pmatrix} 2 & 2 & 6 & 5 & 4 \\ 1 & 3 & 4 & 3 & 5 \\ 5 & 3 & 14 & 12 & 7 \\ 4 & 8 & 14 & 11 & 14 \end{pmatrix}$; **4.8.** $\begin{pmatrix} 1 & 2 & 3 & -3 & 4 \\ 4 & -1 & -2 & -1 & -4 \\ -1 & 7 & 11 & -8 & 16 \\ 9 & 0 & -1 & -5 & -4 \end{pmatrix}$;

4.9. $\begin{pmatrix} 1 & 1 & 3 & 2 & 3 \\ 5 & -3 & 6 & 6 & -3 \\ -2 & 6 & 3 & 0 & 12 \\ 11 & -5 & 15 & 14 & -3 \end{pmatrix}$; **4.10.** $\begin{pmatrix} 1 & 6 & 5 & 4 & -2 \\ 5 & 0 & 2 & -1 & -4 \\ -1 & 7 & 11 & -8 & 16 \\ 9 & 0 & -1 & -5 & -4 \end{pmatrix}$;

4.11. $\begin{pmatrix} 2 & 4 & 1 & -1 & -1 \\ 3 & 1 & 1 & 5 & 5 \\ 3 & 11 & 2 & -8 & -8 \\ 8 & 6 & 3 & 9 & 9 \end{pmatrix}$;

4.12. $\begin{pmatrix} 1 & -2 & 6 & 6 & 4 \\ 3 & -3 & 1 & -2 & 4 \\ 0 & -3 & 17 & 20 & 8 \\ 7 & -8 & 8 & 2 & 12 \end{pmatrix}$;

4.13. $\begin{pmatrix} 3 & -4 & -2 & -4 & 2 \\ 1 & 6 & -1 & 0 & -3 \\ 8 & -18 & -5 & -12 & 9 \\ 5 & 8 & -4 & -4 & -4 \end{pmatrix}$;

4.14. $\begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 3 & 3 & 1 & 3 & -1 \\ 0 & -6 & 8 & -6 & 1 \\ 7 & 5 & 5 & 5 & -2 \end{pmatrix}$;

4.15. $\begin{pmatrix} 1 & 2 & 3 & -2 & 5 \\ 4 & 0 & -1 & -1 & 4 \\ -1 & 6 & 10 & -5 & 11 \\ 9 & 2 & 1 & -4 & 13 \end{pmatrix}$;

4.16. $\begin{pmatrix} 1 & 4 & 3 & 6 & 4 \\ 5 & 2 & -2 & -4 & 2 \\ -2 & 10 & 11 & 22 & 10 \\ 11 & 8 & -1 & -2 & 8 \end{pmatrix}$;

4.17. $\begin{pmatrix} 2 & 0 & 6 & -4 & 2 \\ 2 & -2 & 5 & 5 & -1 \\ 4 & 2 & 13 & -17 & 7 \\ 6 & -4 & 16 & 6 & 0 \end{pmatrix}$;

4.18. $\begin{pmatrix} 4 & 3 & 3 & -4 & -3 \\ 3 & -1 & -1 & 6 & 0 \\ 9 & 10 & 10 & -18 & -9 \\ 10 & 1 & 1 & 8 & -3 \end{pmatrix}$;

4.19. $\begin{pmatrix} 5 & -2 & -2 & 1 & 3 \\ 4 & 5 & 3 & -3 & -1 \\ 11 & -11 & -9 & 6 & 10 \\ 13 & 8 & 4 & -5 & 1 \end{pmatrix}$;

4.20. $\begin{pmatrix} 1 & -1 & 5 & 4 & -4 \\ 5 & 2 & -1 & 6 & 4 \\ -2 & -5 & 16 & 6 & -16 \\ 11 & 3 & 3 & 16 & 4 \end{pmatrix}$;

4.21. $\begin{pmatrix} 4 & 2 & -1 & 2 & -3 \\ 3 & -4 & 0 & 3 & 6 \\ 9 & 10 & -3 & 3 & -15 \\ 10 & -6 & -1 & 8 & 9 \end{pmatrix}$;

4.22. $\begin{pmatrix} 4 & 2 & 1 & 6 & -3 \\ 2 & 0 & 2 & -1 & -1 \\ 10 & 6 & 1 & 19 & -8 \\ 8 & 2 & 5 & 4 & -5 \end{pmatrix}$;

4.23. $\begin{pmatrix} 1 & -3 & 5 & 5 & -2 \\ 2 & 2 & 1 & 6 & -1 \\ 1 & -11 & 14 & 9 & -5 \\ 5 & 1 & 7 & 17 & -4 \end{pmatrix}$;

4.24. $\begin{pmatrix} 2 & 6 & -1 & -1 & 2 \\ 5 & -4 & -1 & -4 & -1 \\ 1 & 22 & -2 & 1 & 7 \\ 12 & -2 & -3 & -9 & 0 \end{pmatrix}$;

4.25. $\begin{pmatrix} 1 & 2 & 5 & -4 & 4 \\ 2 & 3 & -2 & -1 & 0 \\ 1 & 3 & 17 & -11 & 12 \\ 5 & 8 & 1 & -6 & 4 \end{pmatrix}$;

4.26. $\begin{pmatrix} 4 & 0 & 6 & 1 & -4 \\ 5 & 5 & 5 & 0 & -2 \\ 7 & -5 & 13 & 3 & -10 \\ 14 & 10 & 16 & 1 & -8 \end{pmatrix}$;

4.27. $\begin{pmatrix} 3 & -3 & 4 & -3 & -3 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & -12 & 7 & -11 & -10 \\ 11 & 3 & 14 & 1 & -1 \end{pmatrix}$;

4.28. $\begin{pmatrix} 3 & 0 & 2 & -3 & -4 \\ 3 & -3 & 2 & 0 & 3 \\ 6 & 3 & 4 & -9 & -15 \\ 9 & -6 & 6 & -3 & 2 \end{pmatrix}$;

4.29. $\begin{pmatrix} 3 & 5 & 4 & 0 & 5 \\ 1 & 5 & -2 & -3 & -4 \\ 8 & 10 & 14 & 3 & 19 \\ 5 & 15 & 0 & -6 & -3 \end{pmatrix}$;

4.30. $\begin{pmatrix} 1 & 1 & 1 & -2 & 5 \\ 1 & -2 & 3 & 4 & 3 \\ 2 & 5 & 0 & -10 & 12 \\ 3 & -3 & 7 & 6 & 11 \end{pmatrix}$.

Task 5. Solve the following matrix equation by means of the inverse matrix:

5.1. $\begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} \cdot X = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$;

5.2. $X \cdot \begin{pmatrix} 2 & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$;

5.3. $\begin{pmatrix} 2 & -2 \\ 4 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix}$;

5.4. $X \cdot \begin{pmatrix} 0 & -2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$;

5.5. $\begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix}$;

5.6. $X \cdot \begin{pmatrix} 1 & -2 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 1 & 0 \end{pmatrix}$;

5.7. $\begin{pmatrix} 1 & -2 \\ 1 & 5 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$;

5.8. $X \cdot \begin{pmatrix} 7 & 3 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$;

5.9. $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$;

5.10. $X \cdot \begin{pmatrix} 2 & -2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$;

5.11. $\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 5 \\ -1 & -2 \end{pmatrix}$;

5.12. $X \cdot \begin{pmatrix} 4 & -2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$;

5.13. $\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} \cdot X = \begin{pmatrix} 0 & 5 \\ 1 & -3 \end{pmatrix}$;

5.14. $X \cdot \begin{pmatrix} 5 & -2 \\ 4 & -5 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 1 & 1 \end{pmatrix}$;

5.15. $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} -4 & 5 \\ 1 & -3 \end{pmatrix}$;

5.16. $X \cdot \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 1 & -3 \end{pmatrix}$;

5.17. $\begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$;

5.18. $X \cdot \begin{pmatrix} 1 & 3 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 1 & 1 \end{pmatrix}$;

5.19. $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 10 & 5 \\ 1 & -3 \end{pmatrix}$;

5.20. $X \cdot \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 1 & 5 \end{pmatrix}$;

5.21. $\begin{pmatrix} 1 & 4 \\ -3 & 5 \end{pmatrix} \cdot X = \begin{pmatrix} -2 & 3 \\ 1 & -3 \end{pmatrix}$;

5.22. $X \cdot \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$;

5.23. $\begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$;

5.24. $X \cdot \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$;

5.25. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$;

5.26. $X \cdot \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 1 & -3 \end{pmatrix}$;

5.27. $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix}$;

5.28. $X \cdot \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 1 & 2 \end{pmatrix}$;

5.29. $\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 5 \\ 1 & -4 \end{pmatrix}$;

5.30. $X \cdot \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$.

Task 6. Find a solution of the system by means of three different methods, namely: a) by the rule by Cramer; b) by means of the inverse matrix; c) by the method by Gauss.

6.1. $\begin{cases} x_1 + x_2 - 2x_3 = -1, \\ 2x_1 + 3x_2 - 7x_3 = -5, \\ 5x_1 + 2x_2 + x_3 = 6. \end{cases}$

6.2. $\begin{cases} x_1 - 2x_2 + 3x_3 = -1, \\ 2x_1 + 3x_2 - 4x_3 = 5, \\ 3x_1 - 2x_2 - 5x_3 = 1. \end{cases}$

6.3. $\begin{cases} 2x_1 - x_2 + 3x_3 = 6, \\ x_1 + 3x_2 - 2x_3 = -4, \\ 2x_2 - x_3 = -3. \end{cases}$

6.4. $\begin{cases} 2x_1 + x_2 + 4x_3 = 1, \\ 2x_1 - x_2 - 3x_3 = 3, \\ 3x_1 + 4x_2 - 5x_3 = -1. \end{cases}$

6.5. $\begin{cases} x_1 + 5x_2 - x_3 = 3, \\ 2x_1 - x_2 - x_3 = -4, \\ 3x_1 - 2x_2 + 4x_3 = -1. \end{cases}$

6.6. $\begin{cases} x_1 + 5x_2 - x_3 = 4, \\ 2x_1 - x_2 - x_3 = -2, \\ 3x_1 - 2x_2 + 4x_3 = 2. \end{cases}$

6.7. $\begin{cases} 7x_1 + 2x_2 + 3x_3 = 8, \\ 5x_1 - 3x_2 + 2x_3 = 10, \\ 10x_1 - 11x_2 + 5x_3 = 26. \end{cases}$

6.8. $\begin{cases} 7x_1 + 5x_2 + 2x_3 = 5, \\ x_1 - x_2 - x_3 = 2, \\ x_1 + x_2 + 2x_3 = -1. \end{cases}$

6.9. $\begin{cases} 5x_1 + 8x_2 - x_3 = 6, \\ x_1 + 2x_2 + 3x_3 = -2, \\ 2x_1 - 3x_2 + 2x_3 = 0. \end{cases}$

6.10. $\begin{cases} x_1 + 2x_2 + x_3 = 4; \\ 3x_1 + 5x_2 - 7x_3 = 1; \\ 2x_1 + 7x_2 - x_3 = 8. \end{cases}$

6.11. $\begin{cases} 3x_1 + 2x_2 + x_3 = -1, \\ 2x_1 + 3x_2 + x_3 = 1, \\ 2x_1 + x_2 + 3x_3 = -1. \end{cases}$

6.12. $\begin{cases} x_1 + 2x_2 + 4x_3 = 3, \\ 5x_1 + x_2 + 2x_3 = -3, \\ 3x_1 - x_2 + x_3 = -2. \end{cases}$

6.13. $\begin{cases} 4x_1 - 3x_2 + 2x_3 = 10, \\ 2x_1 + 5x_2 - 3x_3 = 1, \\ 5x_1 + 6x_2 - 2x_3 = 8. \end{cases}$

6.14. $\begin{cases} 2x_1 - x_2 - x_3 = 0, \\ 3x_1 + 4x_2 - 2x_3 = 11, \\ 3x_1 - 2x_2 + 4x_3 = -1. \end{cases}$

6.15. $\begin{cases} x_1 + x_2 + 2x_3 = 3, \\ 2x_1 - x_2 + 2x_3 = 4, \\ 4x_1 + x_2 + 4x_3 = 8. \end{cases}$

6.16. $\begin{cases} 3x_1 - x_2 = 3, \\ -2x_1 + x_2 + x_3 = -3, \\ 2x_1 - x_2 + 4x_3 = -2. \end{cases}$

6.17. $\begin{cases} 3x_1 - x_2 + x_3 = 5, \\ 2x_1 - 5x_2 - 3x_3 = -4, \\ x_1 + x_2 - x_3 = -1. \end{cases}$

6.18. $\begin{cases} x_1 + x_2 + x_3 = 1, \\ 2x_1 - x_2 - 6x_3 = 7, \\ 3x_1 - 2x_2 = 1. \end{cases}$

6.19. $\begin{cases} 2x_1 + x_2 - x_3 = 2, \\ x_1 + x_2 + x_3 = 0, \\ 3x_1 - x_2 + x_3 = -2. \end{cases}$

6.20. $\begin{cases} 7x_1 + 5x_2 + 2x_3 = 1, \\ x_1 - x_2 - x_3 = 4, \\ x_1 + x_2 + 2x_3 = -5. \end{cases}$

6.21. $\begin{cases} 4x_1 - 3x_2 + x_3 = 7, \\ -3x_1 - 2x_2 + 2x_3 = -1, \\ -x_1 + x_2 + 5x_3 = -2. \end{cases}$

6.22. $\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1, \\ 3x_1 + 4x_2 - x_3 = 6, \\ -x_1 + x_2 - 2x_3 = -2. \end{cases}$

6.23. $\begin{cases} 3x_1 - 4x_2 - 5x_3 = 1, \\ 3x_1 - x_2 - 5x_3 = 4, \\ 6x_1 + 2x_2 = 2. \end{cases}$

6.24. $\begin{cases} 3x_1 - x_2 + 2x_3 = 0, \\ 2x_1 + x_2 - 3x_3 = 6, \\ x_1 \qquad - x_3 = 2. \end{cases}$

6.25. $\begin{cases} -x_1 + 4x_2 = 2, \\ x_1 - 3x_2 - x_3 = -2, \\ x_1 + 2x_2 + 4x_3 = 8. \end{cases}$

6.26. $\begin{cases} -2x_1 + 2x_2 - 4x_3 = 2, \\ 3x_1 - x_2 = 1, \\ -5x_1 + 6x_2 - 2x_3 = 7. \end{cases}$

6.27. $\begin{cases} 2x_1 - 2x_2 + 2x_3 = -2, \\ 2x_1 + x_2 + x_3 = 2, \\ x_1 + x_2 + 5x_3 = -3. \end{cases}$

6.28. $\begin{cases} 4x_1 - x_2 + 2x_3 = 2, \\ -x_1 + 2x_2 = 3, \\ x_1 - 3x_2 - 5x_3 = -5. \end{cases}$

6.29. $\begin{cases} 3x_1 + 2x_2 + x_3 = 2, \\ -x_1 + 2x_3 = 1, \\ -2x_1 + 2x_2 - 3x_3 = -7. \end{cases}$

6.30. $\begin{cases} 4x_1 - x_2 + 3x_3 = 1, \\ 3x_1 + 2x_2 - 2x_3 = 2, \\ 2x_1 \qquad + x_3 = 1. \end{cases}$

Task 7. Investigate these systems on compatibility and find the solutions of compatible systems. For the homogeneous systems find FSS (fundamental system of solutions).

7.1. a) $\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 0, \\ 3x_1 + x_2 + 2x_3 - x_4 = 4, \\ 2x_1 + 3x_2 - x_3 + 3x_4 = -1, \\ 4x_1 + 2x_2 + 2x_3 + x_4 = 3. \end{cases}$

b) $\begin{cases} x_1 + 3x_2 + 5x_3 - 2x_4 = 0, \\ 2x_1 + 7x_2 + 3x_3 + x_4 = 0, \\ x_1 + 5x_2 + 9x_3 + 8x_4 = 0, \\ 5x_1 + 18x_2 - 4x_3 + 5x_4 = 0. \end{cases}$

7.2. a) $\begin{cases} x_1 \qquad + 2x_3 - x_4 = 1, \\ 3x_1 + x_2 + 6x_3 - 3x_4 = 2, \\ 4x_1 - x_2 + 8x_3 - 4x_4 = 5, \\ 2x_1 + x_2 + 4x_3 - 2x_4 = 1. \end{cases}$

b) $\begin{cases} x_1 + 2x_2 - x_3 + 2x_4 - x_5 = 0, \\ 5x_1 + 11x_2 - 2x_3 + 8x_4 + 2x_5 = 0, \\ 4x_1 + 10x_2 + 2x_3 + 10x_4 - 3x_5 = 0, \\ 5x_1 + 13x_2 + 4x_3 + 16x_4 - 10x_5 = 0. \end{cases}$

7.3. a) $\begin{cases} 3x_1 - 2x_2 + x_3 \qquad = 4, \\ 2x_1 + x_2 - x_3 - 4x_4 = 1, \\ 3x_1 - x_2 + 2x_3 \qquad = 5, \\ x_1 + 2x_2 + x_3 - 2x_4 = 2. \end{cases}$

b) $\begin{cases} 3x_1 - 2x_2 + x_3 - x_4 + 2x_5 = 0, \\ x_1 + x_2 \qquad + 2x_4 + 3x_5 = 0, \\ -2x_1 + 3x_2 - x_3 + 3x_4 + x_5 = 0, \\ 2x_1 - x_2 + 4x_3 - 3x_4 + 4x_5 = 0, \\ -x_1 + x_2 - 3x_3 + 2x_4 + x_5 = 0. \end{cases}$

7.4. a) $\begin{cases} 2x_1 + x_2 + 2x_3 - 2x_4 - 3x_5 = 3, \\ x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 6, \\ 3x_1 - x_2 + 5x_3 - 3x_4 - x_5 = 9. \end{cases}$ **b)** $\begin{cases} x_1 + 2x_2 - 3x_3 + 3x_4 = 0, \\ 2x_1 - x_2 + x_3 - 2x_4 = 0, \\ 3x_1 + x_2 - 2x_3 + x_4 = 0, \\ -x_1 + 3x_2 - x_3 + 2x_4 = 0. \end{cases}$

7.5. a) $\begin{cases} 3x_1 + x_2 - 2x_3 + 4x_4 - 5x_5 = 1, \\ x_1 - 3x_2 + x_3 - 2x_4 + 2x_5 = 2, \\ x_1 + 7x_2 - 4x_3 + 8x_4 - 5x_5 = -3, \\ 5x_1 + 5x_2 - 5x_3 + 10x_4 = 0. \end{cases}$ **b)** $\begin{cases} 2x_1 + 3x_2 - 3x_3 + 2x_4 = 0, \\ 3x_1 + 2x_2 - x_3 + 3x_4 = 0, \\ 2x_1 - x_2 + 5x_3 - x_4 = 0, \\ x_1 + 3x_2 - 6x_3 + 4x_4 = 0. \end{cases}$

7.6. a) $\begin{cases} 9x_1 - 3x_2 + 5x_3 + 6x_4 + x_5 = 4, \\ 6x_1 - 2x_2 + 3x_3 + 4x_4 + 2x_5 = 3, \\ 3x_1 - x_2 + 3x_3 + 14x_4 - x_5 = 0. \end{cases}$ **b)** $\begin{cases} 2x_1 + 3x_2 - 3x_3 + 2x_4 = 0, \\ 3x_1 + 2x_2 - x_3 + 3x_4 = 0, \\ 2x_1 - x_2 + 5x_3 - x_4 = 0, \\ x_1 - x_2 + 2x_3 + x_4 = 0. \end{cases}$

7.7. a) $\begin{cases} x_1 + x_2 - 3x_3 = 3, \\ x_1 - 3x_2 - 3x_3 + 4x_4 = -5, \\ 2x_1 + 3x_2 - 6x_3 - x_4 = 8, \\ 2x_1 - x_2 - 6x_3 + 3x_4 = 0. \end{cases}$ **b)** $\begin{cases} 4x_1 - 3x_2 + 3x_3 = 0, \\ -x_1 + 2x_2 + 3x_3 + 2x_4 = 0, \\ x_1 - 2x_2 + x_3 - x_4 = 0, \\ 3x_1 - x_2 + 2x_3 + x_4 = 0. \end{cases}$

7.8. a) $\begin{cases} 2x_1 + x_2 - x_3 - 3x_4 = 5, \\ 4x_1 + x_3 - 7x_4 = 11, \\ 2x_2 - 3x_3 + x_4 = -1, \\ 2x_1 + 3x_2 - 4x_3 - 2x_4 = 4. \end{cases}$ **b)** $\begin{cases} 3x_1 + x_2 - 2x_3 + x_4 = 0, \\ -x_1 + 3x_2 - x_3 + 2x_4 = 0, \\ 2x_1 + x_2 - 3x_3 + 3x_4 = 0, \\ x_1 + 2x_2 - 3x_3 + x_4 = 0. \end{cases}$

7.9. a) $\begin{cases} 3x_1 - 2x_2 - 3x_3 + 3x_4 = 0, \\ x_1 + 2x_2 - 2x_3 - 2x_4 = 1, \\ x_1 - x_2 - 2x_3 + x_4 = 1, \\ 2x_1 + x_2 - 4x_3 - x_4 = 2. \end{cases}$ **b)** $\begin{cases} x_1 + 5x_2 + 3x_3 + 2x_4 = 0, \\ x_1 + x_2 + 2x_3 + 3x_4 = 0, \\ x_1 + 5x_2 + x_3 + 2x_4 = 0, \\ x_1 + 5x_2 + 5x_3 + 2x_4 = 0. \end{cases}$

7.10 .a) $\begin{cases} 2x_1 - 3x_2 + 4x_3 - x_4 + x_5 = 6, \\ x_1 - 2x_2 + 2x_3 + 3x_4 + 2x_5 = 3, \\ 5x_1 - 2x_2 + x_3 + 4x_4 - x_5 = 6. \end{cases}$ **b)** $\begin{cases} x_1 + x_2 - 3x_4 - x_5 = 0; \\ x_1 - x_2 + 2x_3 - x_4 = 0; \\ 4x_1 - 2x_2 + 6x_3 + 3x_4 - 4x_5 = 0; \\ 2x_1 + 4x_2 - 2x_3 + 4x_4 - 7x_5 = 0. \end{cases}$

7.11. a) $\begin{cases} 3x_1 - x_2 - 5x_3 + 5x_4 = 1, \\ 2x_1 + x_2 - 5x_3 = -1, \\ x_1 - 2x_2 + 5x_4 = 2, \\ x_1 + x_2 - 3x_3 - x_4 = -1. \end{cases}$ **b)** $\begin{cases} 3x_1 + 4x_2 - 5x_3 + 7x_4 = 0, \\ 2x_1 - 3x_2 + 3x_3 - 2x_4 = 0, \\ 4x_1 + 11x_2 - 13x_3 + 16x_4 = 0, \\ 7x_1 - 2x_2 + x_3 + 3x_4 = 0. \end{cases}$

7.12. a) $\begin{cases} 2x_1 + x_2 + x_3 + 3x_4 = 3, \\ x_1 + 2x_2 - x_3 + 2x_4 = -3, \\ x_1 + 3x_2 - x_4 = 0, \\ 3x_1 + 4x_2 + 2x_3 = 6. \end{cases}$ **b)** $\begin{cases} x_1 + 3x_2 + 2x_3 + 7x_4 = 0, \\ 2x_1 - x_2 + 3x_3 - 2x_4 = 0, \\ -3x_1 - 5x_2 + 4x_3 + 16x_4 = 0, \\ x_1 + 17x_2 + 4x_3 + 14x_4 = 0. \end{cases}$

7.13. a) $\begin{cases} x_1 + 2x_2 - 3x_3 - 6x_4 = 4, \\ x_1 - 4x_2 - 3x_3 = -8, \\ x_1 - 2x_2 - 3x_3 - 2x_4 = -4, \\ 2x_1 - 5x_2 - 6x_3 - 3x_4 = -10. \end{cases}$ **b)** $\begin{cases} 5x_1 + x_2 + 4x_3 + 2x_4 = 0, \\ 3x_1 + 2x_2 - x_3 + 3x_4 = 0, \\ 2x_1 - x_2 + 5x_3 - x_4 = 0, \\ -x_1 + 2x_2 - 3x_3 + 2x_4 = 0. \end{cases}$

7.14. a) $\begin{cases} 2x_1 + x_2 - x_3 = 7, \\ x_1 - 3x_2 + x_3 - 3x_4 = 0, \\ 2x_2 + x_3 - 2x_4 = 2, \\ 3x_1 - x_3 - x_4 = 9. \end{cases}$ **b)** $\begin{cases} x_1 - 2x_2 + x_3 + x_4 - x_5 = 0; \\ 2x_1 + x_2 - x_3 - x_4 + x_5 = 0; \\ x_1 + 7x_2 - 5x_3 - 5x_4 + 5x_5 = 0; \\ 3x_1 - x_2 - 2x_3 + x_4 - x_5 = 0. \end{cases}$

7.15. a) $\begin{cases} 2x_1 - x_2 + x_3 - 5x_4 = 2, \\ x_1 + 2x_2 - 5x_3 - 2x_4 = -10, \\ 3x_1 - 5x_2 + x_4 = 0, \\ x_1 + x_2 - 3x_3 - 2x_4 = -6. \end{cases}$ **b)** $\begin{cases} x_1 - 4x_2 + 3x_3 + x_4 + 2x_5 = 0, \\ 3x_1 - 2x_2 - x_3 + 5x_4 - x_5 = 0, \\ 2x_1 + 2x_2 - 4x_3 + 4x_4 - 3x_5 = 0. \end{cases}$

7.16. a) $\begin{cases} 2x_1 - x_2 - 2x_4 = 4, \\ x_1 + x_2 - 3x_3 - x_4 = 2, \\ 3x_1 + x_2 - 5x_3 - 3x_4 = 6, \\ x_1 - x_3 - x_4 = 2. \end{cases}$ **b)** $\begin{cases} 2x_1 - 3x_2 + 4x_3 - 3x_4 = 0, \\ 3x_1 - x_2 + 11x_3 - 13x_4 = 0, \\ 4x_1 + 5x_2 - 7x_3 - 2x_4 = 0, \\ 13x_1 - 25x_2 + x_3 + 11x_4 = 0. \end{cases}$

7.17. a) $\begin{cases} x_1 + x_2 + x_3 - 5x_4 = 0, \\ 3x_1 - 4x_3 - x_4 = -4, \\ 2x_1 - 3x_2 - 3x_3 = 0, \\ x_1 + 2x_2 - 2x_3 + x_4 = -4. \end{cases}$ **b)** $\begin{cases} 5x_1 + 6x_2 - 2x_3 + 7x_4 + 4x_5 = 0, \\ 2x_1 + 3x_2 - x_3 + 4x_4 + 2x_5 = 0, \\ 7x_1 + 9x_2 - 3x_3 + 5x_4 + 6x_5 = 0, \\ 5x_1 + 9x_2 - 3x_3 + x_4 + 6x_5 = 0. \end{cases}$

7.18. a) $\begin{cases} 9x_1 - 3x_2 + 5x_3 + 6x_4 = 4, \\ 6x_1 - 2x_2 + 3x_3 + 4x_4 = 5, \\ 3x_1 - x_2 + 3x_3 + 14x_4 = -8. \end{cases}$ **b)** $\begin{cases} 2x_1 + 3x_2 - x_3 + 5x_4 = 0, \\ 3x_1 - x_2 + 2x_3 - 7x_4 = 0, \\ 4x_1 + x_2 - 3x_3 + 6x_4 = 0, \\ x_1 + 2x_2 - 5x_3 + 13x_4 = 0. \end{cases}$

7.19. a) $\begin{cases} 2x_1 + x_2 + 2x_3 - x_4 = 1, \\ 2x_1 - 3x_2 + 2x_3 + 3x_4 = -3, \\ 3x_1 - x_2 + 3x_3 + x_4 = -1, \\ x_1 - x_2 + x_3 + x_4 = -1. \end{cases}$ **b)** $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 0, \\ x_1 - x_2 + 2x_3 - x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + x_4 = 0. \end{cases}$

7.20. a) $\begin{cases} 3x_1 + 2x_2 + x_3 + 4x_4 + 5x_5 = 3, \\ x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 0, \\ x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 3. \end{cases}$ **b)** $\begin{cases} 2x_1 - 4x_2 - 6x_3 + x_4 = 0; \\ 4x_1 - 3x_2 + 2x_3 - 7x_4 = 0; \\ -x_1 + 2x_2 + 3x_3 = 0; \\ -x_1 + 2x_2 + 3x_3 + x_4 = 0. \end{cases}$

7.21. a) $\begin{cases} x_1 + x_2 + x_3 - 3x_4 = -1, \\ 2x_1 - 4x_2 - x_3 = 1, \\ 3x_1 - 5x_2 - x_4 = 0, \\ 4x_1 - 6x_2 - 3x_3 - 2x_4 = 3. \end{cases}$ **b)** $\begin{cases} x_1 + x_2 - 3x_3 + x_4 = 0, \\ 3x_1 + 5x_2 - 5x_3 + 5x_4 = 0, \\ x_1 + x_2 + x_3 + 3x_4 = 0, \\ x_1 + 2x_2 - 3x_3 + x_4 = 0. \end{cases}$

7.22. a) $\begin{cases} 4x_1 - 2x_2 + 3x_3 - 2x_4 = 3, \\ 2x_1 + x_2 + x_3 + x_4 = 5, \\ 2x_1 - 3x_2 + 2x_3 - 3x_4 = -2. \end{cases}$ **b)** $\begin{cases} x_1 - 3x_2 + 4x_3 - 6x_4 = 0, \\ x_1 + 2x_2 - x_3 + 4x_4 = 0, \\ 3x_1 + x_2 + 2x_3 + 2x_4 = 0, \\ 2x_1 - x_2 + 3x_3 - 2x_4 = 0. \end{cases}$

7.23. a) $\begin{cases} 8x_1 + 6x_2 + 5x_3 + 2x_4 = 21, \\ 3x_1 + 3x_2 + 2x_3 + x_4 = 10, \\ 4x_1 + 2x_2 + 3x_3 + x_4 = 8, \\ 3x_1 + 5x_2 + x_3 + x_4 = 15, \\ 7x_1 + 5x_2 + 5x_3 + 2x_4 = 18. \end{cases}$ **b)** $\begin{cases} x_1 - 2x_2 + x_3 + x_4 - x_5 = 0, \\ 2x_1 + x_2 - x_3 - x_4 + x_5 = 0, \\ x_1 + 7x_2 - 5x_3 - 5x_4 + 5x_5 = 0, \\ 3x_1 - x_2 - 2x_3 + x_4 - x_5 = 0. \end{cases}$

7.24. a) $\begin{cases} 2x_1 + 3x_2 + x_3 + 2x_4 = 4, \\ 4x_1 + 8x_2 + 2x_3 + 3x_4 = 5, \\ 2x_1 + 5x_2 + x_3 + x_4 = 1, \\ x_1 - 7x_2 - x_3 + 2x_4 = 7. \end{cases}$ **b)** $\begin{cases} 2x_1 + 3x_2 - x_3 + 5x_4 = 0, \\ 3x_1 + x_2 + 3x_3 - 2x_4 = 0, \\ 4x_1 + x_2 - 3x_3 + 6x_4 = 0, \\ x_1 - 2x_2 + 4x_3 - 7x_4 = 0. \end{cases}$

7.25. a) $\begin{cases} 3x_1 - x_2 - x_3 - 3x_4 = -3, \\ 2x_1 - x_2 \qquad\quad - 2x_4 = -3, \\ x_1 + x_2 - 3x_3 - x_4 = 3, \\ 2x_1 + x_2 - 4x_3 - 2x_4 = 3. \end{cases}$ **b)** $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 0, \\ 3x_1 - 2x_2 + 2x_3 - 3x_4 = 0, \\ 3x_1 + x_2 - x_3 + 2x_4 = 0, \\ 2x_1 - x_2 + x_3 - 3x_4 = 0. \end{cases}$

7.26. a) $\begin{cases} x_1 + x_2 + x_3 \qquad = -2, \\ 2x_1 + x_2 + x_3 - x_4 = -1, \\ 3x_1 + 2x_2 \qquad + x_4 = 3, \\ 3x_1 - 2x_2 \qquad - x_4 = 9. \end{cases}$ **b)** $\begin{cases} 2x_1 - x_2 + x_3 - x_4 = 0, \\ 4x_1 - 2x_2 - 2x_3 + 3x_4 = 0, \\ 2x_1 - x_2 + 5x_3 - 6x_4 = 0, \\ 2x_1 - x_2 - 3x_3 + 4x_4 = 0. \end{cases}$

7.27. a) $\begin{cases} 2x_1 + 3x_2 - 2x_3 - x_4 = 1, \\ x_1 + 2x_2 - x_3 - x_4 = 1, \\ 2x_1 - x_2 - 2x_3 + 2x_4 = -3, \\ x_1 + x_2 - x_3 \qquad = 0. \end{cases}$ **b)** $\begin{cases} 2x_1 + 7x_2 + 3x_3 + x_4 = 0, \\ x_1 + 3x_2 + 5x_3 - 2x_4 = 0, \\ x_1 + 5x_2 - 9x_3 + 8x_4 = 0, \\ 5x_1 + 18x_2 + 4x_3 + 5x_4 = 0. \end{cases}$

7.28. a) $\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 6, \\ 2x_1 \qquad - x_3 - x_4 = -2, \\ 3x_1 + x_2 \qquad - 2x_4 = 0, \\ x_1 - x_2 + 2x_3 - 4x_4 = 4. \end{cases}$ **b)** $\begin{cases} x_1 + 3x_2 + x_3 - x_4 = 0, \\ 2x_1 - 2x_2 + x_4 = 0, \\ 2x_1 + 3x_2 + x_3 - 3x_4 = 0 \\ 3x_1 + 4x_2 - x_3 + 2x_4 = 0. \end{cases}$

7.29. a) $\begin{cases} x_1 + x_2 - x_3 - x_4 = -1, \\ 2x_1 - 3x_2 - 2x_3 + 3x_4 = 8, \\ x_1 - x_2 - x_3 + x_4 = 3, \\ 3x_1 + x_2 - 3x_3 - 2x_4 = -1. \end{cases}$ **b)** $\begin{cases} x_1 - 4x_2 + 4x_3 - 3x_4 + 2x_5 = 0, \\ 5x_1 - 7x_2 \qquad + 7x_4 - 3x_5 = 0, \\ 2x_1 - 3x_2 + x_3 + x_4 \quad + x_5 = 0, \\ x_1 + x_2 \quad - 3x_3 + 4x_4 - x_5 = 0. \end{cases}$

7.30. a) $\begin{cases} 3x_1 - 2x_2 - x_3 + 2x_4 = 7, \\ 2x_1 + 2x_2 - 4x_3 - x_4 = -1, \\ 3x_1 - x_2 - 3x_3 - 4x_4 = -1, \\ 2x_1 + x_2 + 2x_3 + 2x_4 = 8. \end{cases}$ **b)** $\begin{cases} 2x_1 + 3x_2 - x_3 + 2x_4 + x_5 = 0, \\ 8x_1 + x_2 - 5x_2 + 4x_4 - 3x_5 = 0, \\ -3x_1 + x_2 + 2x_3 - x_4 + 2x_5 = 0, \\ 5x_1 + 2x_2 - 3x_3 + 3x_4 - x_5 = 0. \end{cases}$

CHAPTER 2. ANALYTIC GEOMETRY

2.1. Vector Algebra

2.1.1. Vectors. Basic Definitions and Concepts

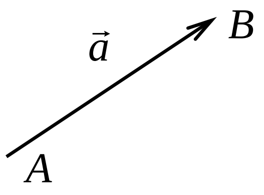


Figure 2

Definition. The vector is a directed segment (Fig.2).

Notations: $\vec{a}$ or $\overline{a}$ or $\overrightarrow{AB}$ or $\overline{AB}$.

Point A is called an origin of the vector.

Point B is called a terminus.

Definition. The distance between the origin and terminus is called a module or a length of this vector.

Notations: $|\vec{a}|$.

If the origin of the vector coincides with the terminus then $|\vec{a}| = 0$. Such a vector is called a zero-vector and denoted as $\vec{0}$ or just 0.

Definition. Two vectors $\vec{a}$ and $\vec{b}$ are called equal if they have the same module and the same direction.

From the last definition It follows that the vectors obtained one from another by parallel shift are equal.

Definition. Two vectors $\vec{a}$ and $\vec{b}$ are called collinear if they are parallel to the same straight line.

Definition. Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called coplanar (or complanar) if they are parallel to the same plane.

Definition. Vector of the unit length having the same direction with vector $\vec{a}$ is called the ort or the unit vector of the vector $\vec{a}$.

Notation: $\vec{a}^{\,o}$.

2.1.2. Linear Operations on Vectors

Linear operations on vectors are the multiplication of vector by scalar and the addition of the vectors.

Definition. The vector $\vec{b} = \lambda\vec{a}$ is called the multiplication of the vector $\vec{a}$ by scalar λ if:

1. $|\vec{b}| = |\lambda||\vec{a}|$;
2. $\vec{a}$ and $\vec{b}$ are collinear vectors;
3. $\vec{a}$ and $\vec{b}$ have the same direction for positive values of λ and the opposite directions for negative values of λ.

See examples on Fig.3.

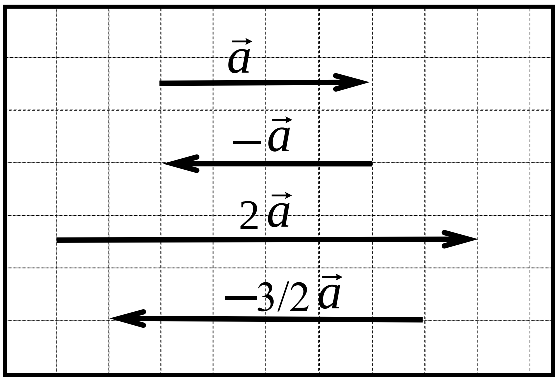


Figure 3

From definition It follows that two collinear vectors could be obtained one from another by multiplication by a scalar. So, we have the following *criterion of collinearity for two nonzero vectors*:

$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda\vec{b} \Leftrightarrow \vec{b} = \mu\vec{a}, \quad \lambda, \mu \in R \setminus \{0\}$$

Definition. The sum of vectors $\vec{a}$ and $\vec{b}$ is called a vector $\vec{c} = \vec{a} + \vec{b}$ which origin coincides with the origin of $\vec{a}$ and terminus coincides with the terminus of $\vec{b}$ if the terminus of $\vec{a}$ and the origin of $\vec{b}$ are connected (Fig.4).

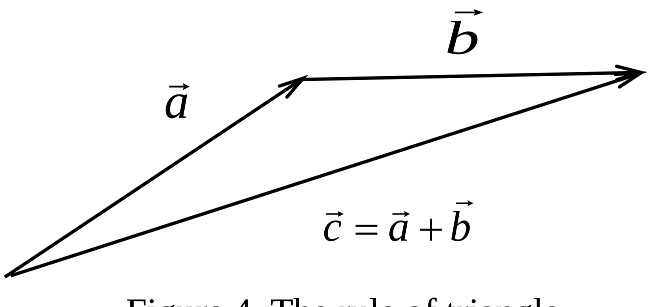


Figure 4. The rule of triangle

This rule to get sum is called the rule of triangle (Fig.4).

There is another rule to get sum called the rule of parallelogram (Fig.5). In this case you should construct a parallelogram on the vectors $\vec{a}$ and $\vec{b}$. The sum of the vectors coincides with the diagonal of this parallelogram directed from the origin of $\vec{a}$ to the terminus of $\vec{b}$.

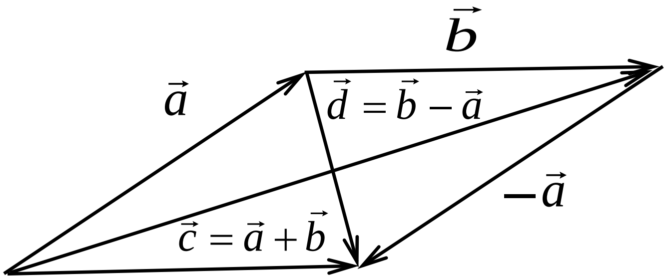


Figure 5. The rule of parallelogram

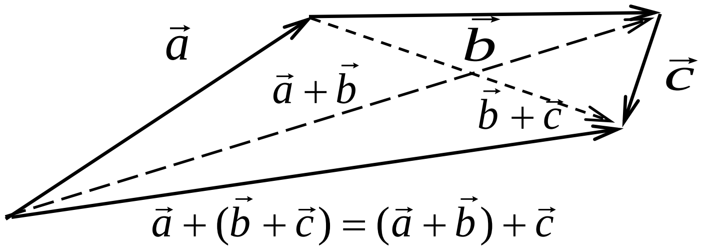


Figure 6

To get difference of vectors you should fulfill the following operations:

$$\vec{d} = \vec{b} - \vec{a} = \vec{b} + (-1)\vec{a} = \vec{b} + (-\vec{a}).$$

The difference of vectors coincides with the other diagonal of the parallelogram constructed on $\vec{a}$ and $\vec{b}$ (Fig.5). It is directed from minuend origin to subtrahend origin if their terminuses are connected.

Basic properties of linear operations:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Fig.6)
3. $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$, $\lambda \in R$
4. $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$, $\lambda, \mu \in R$

Example. Let us find a vector $\vec{c}$ with direction coinciding with a bisector of an angle between the vectors $\vec{a}$ and $\vec{b}$ (Fig.7).

A diagonal bisects the angle of a parallelogram only if this parallelogram is a rhomb. That is why the vector $\vec{c}$ bisects an angle between two vectors only if their lengths are equal to each other.

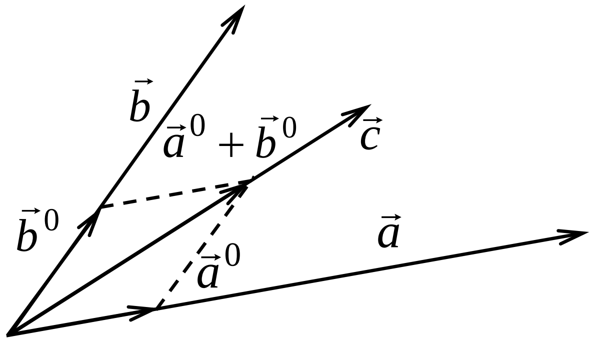


Figure 7

Let us consider the orts $\vec{a}^0$ and $\vec{b}^0$. Their lengths are equal to one and their directions coincide with directions of $\vec{a}$ and $\vec{b}$, relatively. Then the vector directed along the bisector of the angle between $\vec{a}$ and $\vec{b}$ has the same direction as $\vec{a}^0+\vec{b}^0$. Therefore,

$$\vec{c}=\lambda(\vec{a}^0+\vec{b}^0),\ \lambda>0.$$

Note, that there are several other ways to construct the vectors of the equal length. For example, we can find the bisector as

$$\vec{c}=\lambda\left(\vec{a}\cdot\left|\vec{b}\right|+\vec{b}\cdot\left|\vec{a}\right|\right),\ \lambda>0.$$

2.1.3. Concept of Linear Space

Definition. The set L of the elements $x, y, z, \ldots$ is called linear space (LS) if

I. There is an operation of multiplication of elements by scalar such that

$$\forall x \in L \Rightarrow \alpha x \in L \ \ \forall \alpha \in R;$$

II. There is an operation of addition such that

$$\forall x, y \in L \Rightarrow x + y \in L;$$

III. These operations satisfy 8 conditions:

1. $x + y = y + x \quad \forall x, y \in L$;
2. $(x + y) + z = x + (y + z) \quad \forall x, y, z \in L$;

3. $\exists 0 \in L:\ x+0=0+x=x\ \ \forall x \in L$;
4. $\forall x \in L\ \exists y=-x \in L:\ x+y=0$;
5. $1\cdot x = x\ \ \forall x \in L$;
6. $\lambda(x+y)=\lambda x+\lambda y\ \ \forall x,y \in L\ \forall \lambda \in R$;
7. $(\lambda+\mu)x=\lambda x+\mu x\ \ \forall x \in L\ \forall \lambda,\mu \in R$;
8. $(\lambda\mu)x=\lambda(\mu x)=\mu(\lambda x)\ \ \forall x \in L\ \forall \lambda,\mu \in R$.

Example 1. The set of continuous functions on segment $[a,b]$ is a linear space. Indeed, usual operations of addition and multiplication by number satisfy all conditions: the sum of continuous functions is continuous function, continuous function multiplied by a number is still continuous function, zero-element is zero function which is obviously continuous and so on.

Example 2. The set of real numbers is a linear space. But the set of integers is not *LS*, since, for example, multiplication of any integer by real number π makes this number not integer.

Example 3. The set of all matrices of the identical size is a linear space.

Example 4. The set of vectors is a linear space.

Below we are going to consider some properties of linear spaces only on the example of vector space since vectors are objects of our consideration. But these properties are the same for any linear space.

2.1.4. Concept of Basis. Decomposition of the Vector

Definition. The expression $\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\alpha_3\vec{a}_3+\ldots+\alpha_n\vec{a}_n,\ \alpha_1,\alpha_2,\ldots\alpha_n \in R$ is called linear combination (LC) of the vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3,\ldots,\vec{a}_n$.

Definition. The vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3,\ldots,\vec{a}_n$ are called linearly independent (LI) if any their trivial (zero) linear combination has trivial coefficients, i.e.

$\vec{a}_1,\vec{a}_2,\ldots,\vec{a}_n$ are LI $\Leftrightarrow\left(\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\ldots+\alpha_n\vec{a}_n=0\ \ \Rightarrow\ \ \alpha_1=\alpha_2=\ldots=\alpha_n=0\right)$.

In other case they are called linearly dependent (LD), i.e.

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are LD $\Leftrightarrow \exists k : \alpha_k \neq 0$ and $\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 + \ldots + \alpha_n\vec{a}_n = 0$.

Theorem (Linear dependence of vectors) The vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ are linearly dependent if and only if one of these vectors is linear combination of other vectors.

Proof. Necessity. We know that vectors are linearly dependent. We should proof that one of them is linear combination of others. Suppose we have some zero linear combination of vectors. Then at least one coefficient of it is not equal to zero. Suppose it has number *k*, i.e. $\alpha_k \neq 0$. We divide zero expression by $-\alpha_k$ and express from obtained equation the vector $\vec{a}_k$:

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \ldots + \alpha_k\vec{a}_k + \ldots + \alpha_n\vec{a}_n = 0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{\alpha_1}{\alpha_k}\vec{a}_1 - \frac{\alpha_2}{\alpha_k}\vec{a}_2 - \ldots - \vec{a}_k - \ldots - \frac{\alpha_n}{\alpha_k}\vec{a}_n = 0 \Leftrightarrow$$

$$\Leftrightarrow \vec{a}_k = -\frac{\alpha_1}{\alpha_k}\vec{a}_1 - \frac{\alpha_2}{\alpha_k}\vec{a}_2 - \ldots - \frac{\alpha_{k-1}}{\alpha_k}\vec{a}_{k-1} - \frac{\alpha_{k+1}}{\alpha_k}\vec{a}_{k+1} - \ldots - \frac{\alpha_m}{\alpha_k}\vec{a}_m = 0 \Leftrightarrow$$

$$\Leftrightarrow \vec{a}_k = -\sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{\alpha_i}{\alpha_k}\vec{a}_i .$$

Necessity is proven.

Sufficiency. Suppose $\vec{a}_k = \sum\limits_{\substack{i=1 \\ i\neq k}}^{n} \gamma_i \vec{a}_i$. We should prove that vectors are linearly dependent. Let us put $\vec{a}_k$ to the right of the last equation. So we get

$$0 = \sum_{\substack{i=1 \\ i\neq k}}^{n} \gamma_i \vec{a}_i - \vec{a}_k ,$$

i.e. we have obtained zero linear combination of all vectors with the coefficient $\gamma_k = -1 \neq 0$. From definition of linear dependence it means that these vectors are LD. ***Theorem is proven.***

By means of this Theorem we will prove the following 3 statements:

Statement 1. Two vectors are linearly dependent if and only if they are collinear.

Proof. $\vec{a},\vec{b}$ are LD $\Leftrightarrow$ [by Theorem 1] $\Leftrightarrow \vec{a}=\alpha\vec{b} \Leftrightarrow \vec{a},\vec{b}$ are collinear.

Statement is proven.

Corollary. Two vectors are linearly independent if and only if they are not collinear.

Statement 2. Three vectors are linearly dependent if and only if they are coplanar.

Proof. Necessity. $\vec{a},\vec{b},\vec{c}$ are LD. By the Theorem 1 we get, for example, that $\vec{c}=\alpha\vec{a}+\beta\vec{b}$. Thus, $\vec{c}$ is a diagonal of the parallelogram constructed on $\alpha\vec{a}$ and $\beta\vec{b}$ and it belongs to the plane of this parallelogram as $\vec{a}$ and $\vec{b}$ do. So these vectors are coplanar.

To prove Sufficiency we need just to prove that for any three coplanar vectors one is linear combination of others.

Suppose $\vec{a},\vec{b}$ are collinear. Then

$$\vec{a}=\lambda\vec{b} \text{ or } \vec{a}=\lambda\vec{b}+0\cdot\vec{c},$$

so vectors are LD by Theorem 1. Suppose now that $\vec{a},\vec{b}$ are not collinear. Then, accordingly to the Fig.8,

$$\vec{c}=\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{AC}=\lambda\vec{a}+\mu\vec{b},$$

since *ABDC* is a parallelogram and $\overrightarrow{AB}\,||\,\vec{a}$, $\overrightarrow{AC}\,||\,\vec{b}$. ***Statement is proven.***

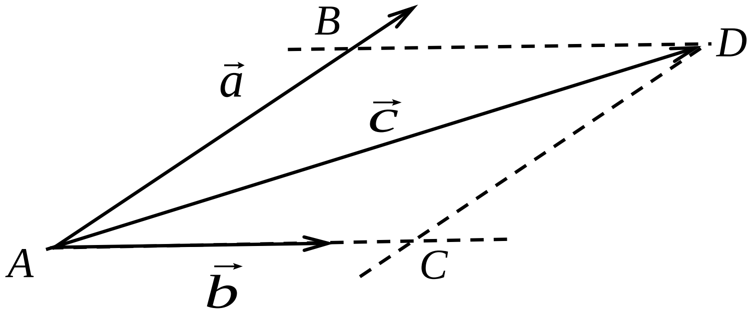


Figure 8

Statement 3. Any four vectors in space are linearly dependent.

Proof. Let us consider any four vectors in space. There are two cases.

Case 1. $\vec{a},\vec{b},\vec{c}$ are coplanar. Then they are LD, i.e. $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}+0\cdot\vec{d}=0$ with not all zero coefficients. Thus $\vec{a},\vec{b},\vec{c},\vec{d}$ are LD as well.

Case 2. $\vec{a},\vec{b},\vec{c}$ are not coplanar. Then let us draw a straight line through the terminus of the vector $\vec{d}$ (point *F*) parallel to the vector $\vec{c}$ to find point *D* which is an intersection of constructed straight line and plane of the vectors $\vec{a},\vec{b}$ (Fig.9). Obtained vector $\overrightarrow{AD}$ is coplanar with not collinear vectors $\vec{a},\vec{b}$, i.e. it could be presented as their linear combination. In other case, vector $\overrightarrow{DF}=\overrightarrow{AE}$ is collinear to $\vec{c}$, i.e.

$$\overrightarrow{DF}=\lambda\vec{c},\ \lambda\neq 0.$$

So,

$$\vec{d}=\overrightarrow{AF}=\overrightarrow{AE}+\overrightarrow{AD}=\overrightarrow{AE}+\overrightarrow{AB}+\overrightarrow{AC}=\lambda\vec{c}+\alpha\vec{a}+\beta\vec{b},$$

where $\lambda\neq 0$. It means that these vectors are linearly dependent. ***Statement is proven.***

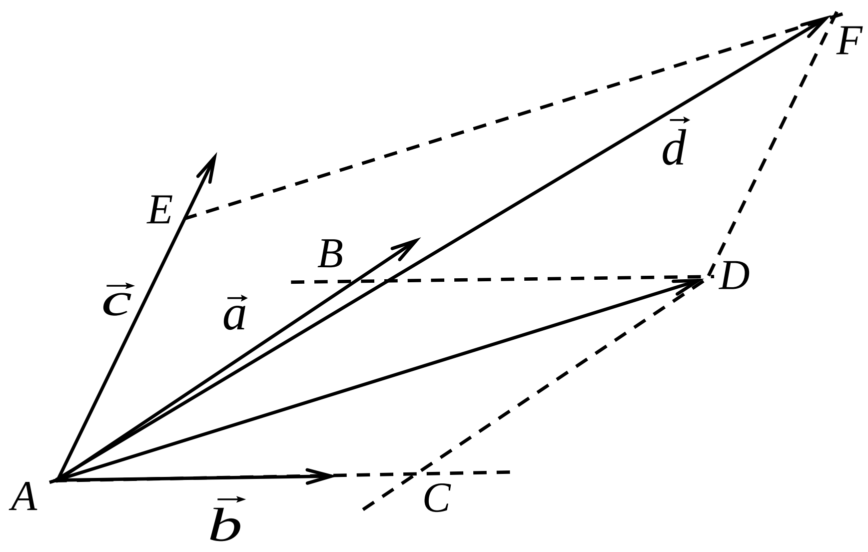


Figure 9

Definition. Linear space L is called n-dimensional if there are n linearly independent elements and any $(n+1)$ are linearly dependent.

Definition. In n-dimensional linear space any n linearly independent elements are called basis of this space.

Note 1. There could be a lot of different bases in the same *LS*.

Note 2. From statements 1-3 follows that:

1. Plane is 2-dimensional *LS* and any two not collinear vectors form basis.
2. Space is 3-dimensional *LS* and any three not coplanar vectors form basis.

Definition. The basis is called orthogonal if every two vectors from basis are perpendicular to each other.

Definition. The basis is called orthonormal if it is orthogonal and the module of every vector is equal to 1.

Theorem (Decomposition of the vector in the basis) Any vector of n-dimensional linear space can be presented as linear combination of basic vectors and this presentation is unique.

Proof. Let us consider any basis of the n-dimensional linear space $\vec{e}_1,\vec{e}_2,\vec{e}_3,\ldots\vec{e}_n$ and an arbitrary vector $\vec{x}$. From definition of the n-dimensional linear space it follows that vectors $\vec{x},\vec{e}_1,\vec{e}_2,\vec{e}_3,\ldots\vec{e}_n$ are linearly dependent, i.e. we have some zero linear combination of these vectors

$$\gamma_0\vec{x}+\gamma_1\vec{e}_1+\gamma_2\vec{e}_2+\gamma_3\vec{e}_3+\ldots+\gamma_n\vec{e}_n=0$$

with not all zero coefficients. There are two possible cases.

Case 1. $\gamma_0=0$. Then we found zero linear combination of the basis vectors with not all zero coefficients: $\gamma_1\vec{e}_1+\gamma_2\vec{e}_2+\gamma_3\vec{e}_3+\ldots+\gamma_n\vec{e}_n=0$. It means that basis vectors are LD. We got a contradiction with definition of basis.

Case 2. $\gamma_0\neq 0$. So we got the presentation of the vector through basis ones:

$$\vec{x}=-\frac{\gamma_1}{\gamma_0}\vec{e}_1-\frac{\gamma_2}{\gamma_0}\vec{e}_2-\frac{\gamma_3}{\gamma_0}\vec{e}_3-\ldots-\frac{\gamma_n}{\gamma_0}\vec{e}_n.$$

Let us prove that this presentation is unique. We suppose the opposite statement, namely, that there are two different presentations:

$$\vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n \text{ and } \vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n .$$

After subtraction of two last equations one from another we have

$$0 = (\beta_1 - \alpha_1)\vec{e}_1 + (\beta_2 - \alpha_2)\vec{e}_2 + \ldots + (\beta_n - \alpha_n)\vec{e}_n .$$

Since the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$ are linearly independent, we have

$$\begin{cases} \beta_1 - \alpha_1 = 0, \text{ i.e. } \beta_1 = \alpha_1; \\ \beta_2 - \alpha_2 = 0, \text{ i.e. } \beta_2 = \alpha_1; \\ \qquad\qquad \vdots \\ \beta_n - \alpha_n = 0, \text{ i.e. } \beta_n = \alpha_n. \end{cases}$$

So, any two presentations have the same coefficients, i.e. presentation is unique.***Theorem is proven.***

Definition. The presentation of the vector as linear combination of basic vectors is called the decomposition of the vector $\vec{x}$ in the basis $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$. At the same time the coefficients of this decomposition are called the coordinates of the vector $\vec{x}$ in this basis.

Note 1. If the basis is chosen, one can write only coordinates of vector instead of the whole decomposition, i.e. one can write that $\vec{x} = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ or $\vec{x} = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ instead of $\vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n$. Moreover, both signs "," and ";" might be used to separate coordinates.

Note 2. Suppose we have 2 or 3-dimentional space. From Statements 2 and 3 and the last Theorem It follows the way to find decomposition of the vector in any chosen basis.

Note 3. Since each vector can be associated with row/column of its coordinates, linear dependence/independence of vectors coincides with linear dependence/independence of rows/columns. Thus, to check linear dependence of the vectors given by their coordinates It is enough and sufficient to check it for the rows/columns of their coordinates.

2.1.5. Linear Operations on Vectors Given by Their Coordinates in Some Basis

Suppose we consider some n-dimensional linear space with basis $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$ and vectors

$\vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$,

$\vec{y} = \beta_1\vec{e}_1 + \beta_2\vec{e}_2 + \beta_3\vec{e}_3 + \ldots + \beta_n\vec{e}_n = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n)$.

Then

$$\lambda\vec{x} = \lambda(\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \ldots + \alpha_n\vec{e}_n) = \lambda\alpha_1\vec{e}_1 + \lambda\alpha_2\vec{e}_2 + \ldots + \lambda\alpha_n\vec{e}_n = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \ldots, \lambda\alpha_n).$$

$$\vec{x} + \vec{y} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n + \beta_1\vec{e}_1 + \beta_2\vec{e}_2 + \beta_3\vec{e}_3 + \ldots + \beta_n\vec{e}_n =$$

$$= (\alpha_1 + \beta_1)\vec{e}_1 + (\alpha_2 + \beta_2)\vec{e}_2 + \ldots + (\alpha_n + \beta_n)\vec{e}_n = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3, \ldots, \alpha_n + \beta_n).$$

Conclusions:

1. To multiply vector by scalar means to multiply all its coordinates by this scalar;
2. To add two vectors means to add their corresponding coordinates.

Example. Show that the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form a basis and find decomposition $\vec{b}$ in this given basis if

$$\vec{a}_1 = (1;-1;2), \vec{a}_2 = (2;2;-1), \vec{a}_3 = (2;1;0), \vec{b} = (3;7;-7).$$

To decompose the vector $\vec{b}$ in this basis $\vec{a}_1, \vec{a}_2, \vec{a}_3$ means to find the following its presentation

$$\vec{b} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3.$$

The last equality is equivalent to the following:

$$(3;7;-7) = \alpha_1(1;-1;2) + \alpha_2(2;2;-1) + \alpha_3(2;1;0) =$$
$$= (\alpha_1 + 2\alpha_2 + 2\alpha_3; -\alpha_1 + 2\alpha_2 + \alpha_3; 2\alpha_1 - \alpha_2)$$

or

$$\begin{cases} \alpha_1 + 2\alpha_2 + 2\alpha_3 = 3, \\ -\alpha_1 + 2\alpha_2 + \alpha_3 = 7, \\ 2\alpha_1 - \alpha_2 = -7. \end{cases}$$

Extended matrix of the obtained system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -1 & 2 & 1 & 7 \\ 2 & -1 & 0 & -7 \end{array}\right).$$

Note, that the matrix of this system consists of the columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$. If the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form the basis then they are linearly independent and a rank of the system matrix is equal to 3. Thus we can answer both questions of this example by solving this system. If the rank of system matrix is equal to 3 then the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form basis and solutions of the system are the coordinates of $\vec{b}$ in this basis.

Let us solve this inhomogeneous system of equations relatively to $\alpha_1, \alpha_2, \alpha_3$ by Jordan-Gauss method:

$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -1 & 2 & 1 & 7 \\ 2 & -1 & 0 & -7 \end{array}\right)$~[We add multiplied by 1 to the second row and the first row multiplied by (−2) to the third one] ~ $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 4 & 3 & 10 \\ 0 & -5 & -4 & -13 \end{array}\right)$~[We add the last row to the second one to get 1 in the second row]~

~ $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -1 & -1 & -3 \\ 0 & 4 & 3 & 10 \end{array}\right)$~[We add the multiplied by 2 to the first row and the second row multiplied by 4 to the third one]~ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -2 \end{array}\right)$~[We add the third row to the second and multiply the third row by (−1)] ~ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$.

The rank of the system matrix is equal to 3 and to the rank of the extended matrix. Thus, the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form the basis and the system is compatible. Here $\alpha_1 = -3; \alpha_2 = 1; \alpha_3 = 2.$

Therefore $\vec{b} = -3\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$ is a decomposition of the vector $\vec{b}$ in the basis $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

2.1.6. Transition to a New Basis

It is absolutely clear that coordinates of the vector depend on the chosen basis and vary at transition from one basis to another one.

Suppose we have two different bases: old basis $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$ and new one $\vec{e}_1^{\,*}, \vec{e}_2^{\,*}, \vec{e}_3^{\,*}, \ldots \vec{e}_n^{\,*}$.

Each vector of the new basis could be decomposed in the old basis:

$$\begin{cases} \vec{e}_1^{\,*} = a_{11}\vec{e}_1 + a_{21}\vec{e}_2 + a_{31}\vec{e}_3 + \ldots + a_{n1}\vec{e}_n \\ \vec{e}_2^{\,*} = a_{12}\vec{e}_1 + a_{22}\vec{e}_2 + a_{32}\vec{e}_3 + \ldots + a_{n2}\vec{e}_n \\ \vdots \\ \vec{e}_n^{\,*} = a_{1n}\vec{e}_1 + a_{2n}\vec{e}_2 + a_{3n}\vec{e}_3 + \ldots + a_{nn}\vec{e}_n \end{cases}, \text{ i.e. } \mathrm{E}^* = A^T\mathrm{E}, \text{ where } \mathrm{E} = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ \vec{e}_n \end{pmatrix},$$

$$\mathrm{E}^* = \begin{pmatrix} \vec{e}_1^{\,*} \\ \vec{e}_2^{\,*} \\ \vdots \\ \vec{e}_n^{\,*} \end{pmatrix}.$$

Obtained matrix A is called *the matrix of transition* from old basis to new one.

Note 1. $\det A \neq 0$, since in other case rows of this matrix are linearly dependant, i.e. one row is linear combination of others and thus one vector of new basis is linear combination of other basis vectors. The last statement contradicts with definition of basis.

Note 2. Each column of A is a decomposition of the corresponding new basic vector in old basis.

Let us find the rule of coordinate change at basis transition by means of the matrix A.

$$\vec{x} = \alpha_1^*\vec{e}_1^{\,*} + \alpha_2^*\vec{e}_2^{\,*} + \alpha_3^*\vec{e}_3^{\,*} + \ldots + \alpha_n^*\vec{e}_n^{\,*} =$$
$$= \alpha_1^*(a_{11}\vec{e}_1 + a_{21}\vec{e}_2 + a_{31}\vec{e}_3 + \ldots + a_{n1}\vec{e}_n) + \alpha_2^*(a_{12}\vec{e}_1 + a_{22}\vec{e}_2 + a_{32}\vec{e}_3 + \ldots + a_{n2}\vec{e}_n) +$$
$$+ \alpha_3^*(a_{13}\vec{e}_1 + a_{23}\vec{e}_2 + a_{33}\vec{e}_3 + \ldots + a_{n3}\vec{e}_n) + \ldots + \alpha_n^*(a_{1n}\vec{e}_1 + a_{2n}\vec{e}_2 + \ldots + a_{nn}\vec{e}_n) =$$

$=(a_{11}\alpha_1^* + a_{12}\alpha_2^* + a_{13}\alpha_3^* + \ldots + a_{1n}\alpha_n^*)\vec{e}_1 + (a_{21}\alpha_1^* + a_{22}\alpha_2^* + \ldots + a_{2n}\alpha_n^*)\vec{e}_2 +$
$+(a_{31}\alpha_1^* + a_{32}\alpha_2^* + \ldots + a_{3n}\alpha_n^*)\vec{e}_3 + \ldots + (a_{n1}\alpha_1^* + a_{n2}\alpha_2^* + \ldots + a_{nn}\alpha_n^*)\vec{e}_n =$
$=\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n .$

Thus,

$$\begin{cases} \alpha_1 = a_{11}\alpha_1^* + a_{12}\alpha_2^* + \ldots + a_{1n}\alpha_n^* \\ \alpha_2 = a_{21}\alpha_1^* + a_{22}\alpha_2^* + \ldots + a_{2n}\alpha_n^* \\ \alpha_3 = a_{31}\alpha_1^* + a_{32}\alpha_2^* + \ldots + a_{3n}\alpha_n^* \\ \vdots \\ \alpha_n = a_{n1}\alpha_1^* + a_{n2}\alpha_2^* + \ldots + a_{nn}\alpha_n^* \end{cases} \Leftrightarrow \boxed{\begin{matrix} X = AX^* \\ X^* = A^{-1}X \end{matrix}} \text{ where } X = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},\ X^* = \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{pmatrix}.$$

Example. Find the matrix of transition from the basis $\vec{e}_1 = (1;2;1)$, $\vec{e}_2 = (-1;1;1)$, $\vec{e}_3 = (0;1;0)$ to the basis $\vec{e}_1^{\,*} = (0;4;2)$, $\vec{e}_2^{\,*} = (2;0;0)$, $\vec{e}_3^{\,*} = (0;2;2)$.

To find this matrix we should decompose the vectors of the new basis in the old one, i.e. to solve the following systems of equations:

$$\begin{cases} \alpha_1 - \alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 4 \\ \alpha_1 + \alpha_2 = 2 \end{cases} \qquad \begin{cases} \alpha_1 - \alpha_2 = 2 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \qquad \begin{cases} \alpha_1 - \alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 2 \\ \alpha_1 + \alpha_2 = 2 \end{cases}$$

The matrix system is the same. The only difference is in right parts. That is why we can solve these systems at the same time by means of the modified extended matrix:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 4 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 3 & 1 & 4 & -4 & 2 \\ 0 & 2 & 0 & 2 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 & -2 & 0 \\ 0 & 2 & 0 & 2 & -2 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 & -2 & 0 \\ 0 & 0 & -2 & -2 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right).$$

Thus, the solution of the first system is

$$\alpha_1 = 1; \alpha_2 = 1; \alpha_3 = 1 \text{ and } \vec{e}_1^{\,*} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3;$$

the solution of the second system is

$$\alpha_1 = 1; \alpha_2 = -1; \alpha_3 = -1 \text{ and } \vec{e}_2^{\,*} = \vec{e}_1 - \vec{e}_2 - \vec{e}_3;$$

the solution of the third system is

$$\alpha_1 = 1; \alpha_2 = 1; \alpha_3 = -1 \text{ and } \vec{e}_3^{\,*} = \vec{e}_1 + \vec{e}_2 - \vec{e}_3.$$

Therefore

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

2.1.7. Projection of the Vector on Axis

Let us consider an arbitrary vector $\overrightarrow{AB}$ and an axis with direction given by the vector $\vec{u}$ (Fig.10). To get points A^*, B^* we drop perpendiculars from the origin and terminus of the vector on the axis.

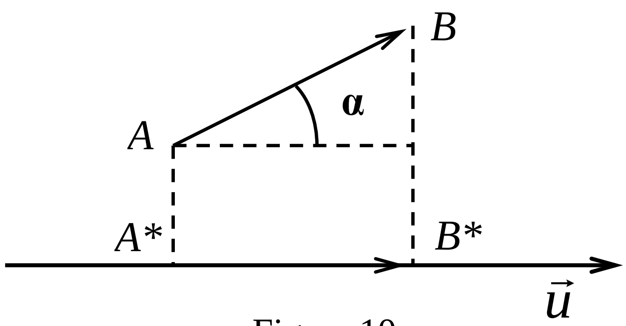


Figure 10

Definition. The length of the segment A^*B^* taken with sigh "+" if $\overrightarrow{A^*B^*}$ has the same direction with $\vec{u}$ or with the sigh "–" if $\overrightarrow{A^*B^*}$ has the opposite direction with $\vec{u}$ is called projection of the vector $\overrightarrow{AB}$ on $\vec{u}$ (or on the axis with direction $\vec{u}$). Notation: $pr_{\vec{u}}\overrightarrow{AB}$.

Note. From the definition and Fig.10 it follows that

$$pr_{\vec{u}}\overrightarrow{AB} = \left|\overrightarrow{A^*B^*}\right| = \left|\overrightarrow{AB}\right| \cos(\widehat{\overrightarrow{AB}, \vec{u}}\,) = \left|\overrightarrow{AB}\right| \cos\alpha\,.$$

Properties of the projections:

1. $pr_{\vec{u}}\lambda\vec{a} = \lambda pr_{\vec{u}}\vec{a}$;
2. $pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}$.
3. $pr_{\lambda\vec{u}}\vec{a} = pr_{\vec{u}}\vec{a}$ for positive λ;

   $pr_{\lambda\vec{u}}\vec{a} = -pr_{\vec{u}}\vec{a}$ for negative λ.

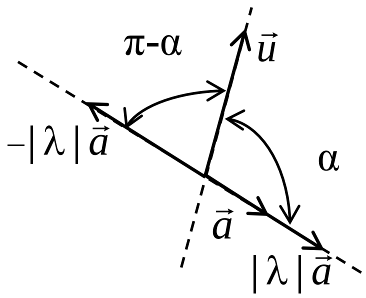


Figure 11

Proof. 1. Let $\alpha = (\widehat{\vec{a},\vec{u}})$ (Fig.11). Then

$$pr_{\vec{u}}\lambda\vec{a} = |\lambda\vec{a}|\cos(\widehat{\lambda\vec{a},\vec{u}}) = |\lambda||\vec{a}|\cos(\widehat{\lambda\vec{a},\vec{u}}) =$$

$$= \begin{cases} |\lambda||\vec{a}|\cos\alpha \ \ if \ \ \lambda \ge 0 \\ |\lambda||\vec{a}|\cos(\pi-\alpha) \ \ if \ \ \lambda < 0 \end{cases} = \begin{cases} |\lambda||\vec{a}|\cos\alpha \ \ if \ \ \lambda \ge 0 \\ -|\lambda||\vec{a}|\cos\alpha \ \ if \ \ \lambda < 0 \end{cases} = \lambda|\vec{a}|\cos\alpha = \lambda pr_{\vec{u}}\vec{a}.$$

2. Let us prove this property geometrically. There are six different cases (Fig.12).

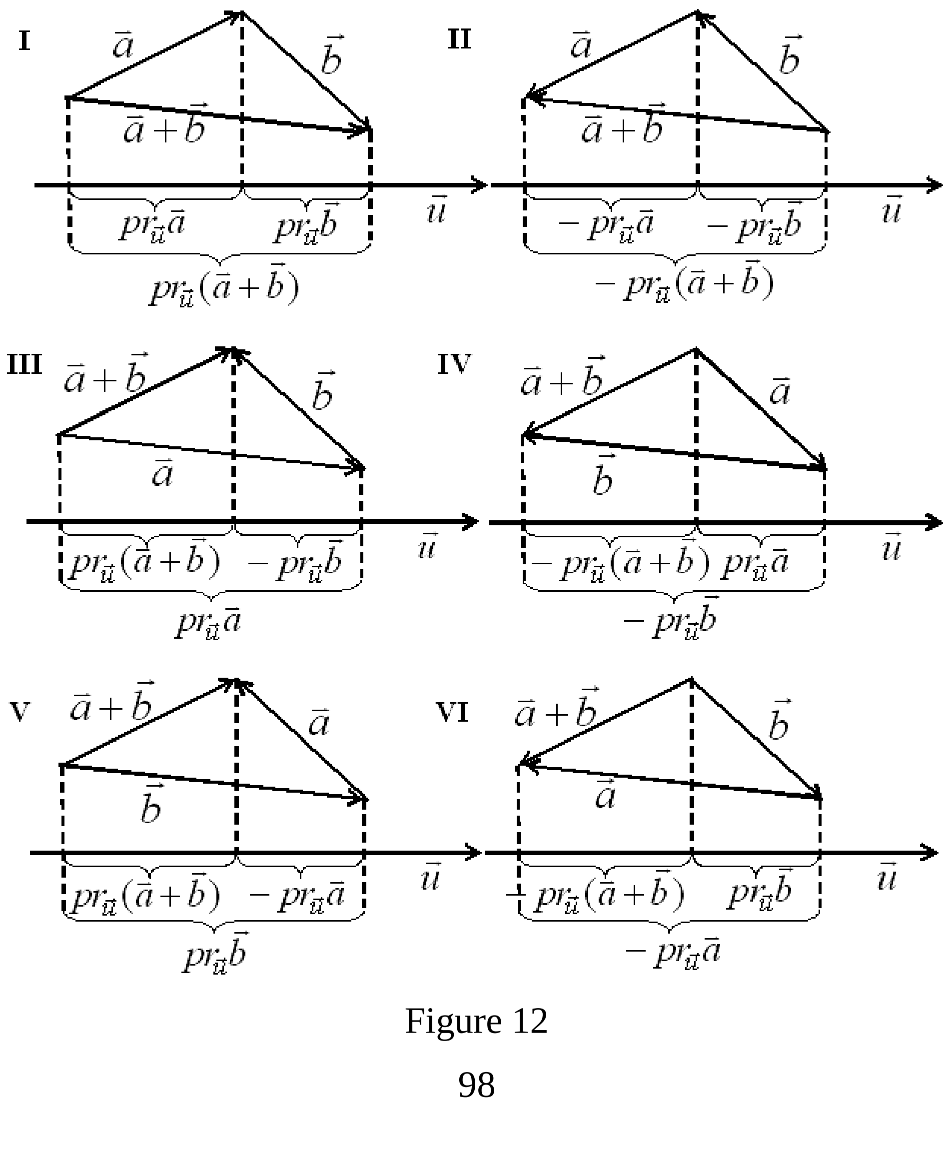


Figure 12

It follows from case I that

$$pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\,.$$

It follows from case II that

$$-pr_{\vec{u}}(\vec{a}+\vec{b}) = -pr_{\vec{u}}\vec{a} + -pr_{\vec{u}}\vec{b}\,,\ \text{i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\,.$$

It follows from case III that

$$pr_{\vec{u}}\vec{a} = pr_{\vec{u}}(\vec{a}+\vec{b}) - pr_{\vec{u}}\vec{b}\,,\ \text{i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\,.$$

It follows from case IV that

$$-pr_{\vec{u}}\vec{b} = -pr_{\vec{u}}(\vec{a}+\vec{b}) + pr_{\vec{u}}\vec{a}\,,\ \text{i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\,.$$

It follows from case V that

$$pr_{\vec{u}}\vec{b} = pr_{\vec{u}}(\vec{a}+\vec{b}) - pr_{\vec{u}}\vec{a}\,,\ \text{i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\,.$$

It follows from case VI that

$$-pr_{\vec{u}}\vec{a} = -pr_{\vec{u}}(\vec{a}+\vec{b}) + pr_{\vec{u}}\vec{b}\,,\ \text{i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\,.$$

3. Since for the positive λ the direction of the axis stays the same, the projection of the vector saves its value. For the negative λ we obtain the opposite direction of the axis and therefore the opposite sign of the projection.

Properties are proven.

Note. One additional property of vector projection follows directly from the definition:

Example. It is known that $pr_{\vec{c}}\vec{a} = 10$, $pr_{\vec{c}}\vec{b} = 5$. Find $pr_{-\vec{c}}(3\vec{a} - 2\vec{b})$.

By the projection properties we have

$$pr_{-\vec{c}}(3\vec{a}-2\vec{b}) = -pr_{\vec{c}}(3\vec{a}-2\vec{b}) = -3pr_{\vec{c}}\vec{a} + 2pr_{\vec{c}}\vec{b} = -3\cdot 10 + 2\cdot 5 = -30 + 10 = -20\,.$$

Thus, the vector $-\vec{c}$ and the vector-projection of the vector $3\vec{a} - 2\vec{b}$ have the opposite directions.

2.1.8. Cartesian Coordinate System

Cartesian coordinate system consists of a point *O* called the origin and perpendicular directed coordinate axes passing through the origin.

Cartesian coordinate system with two (three) axes is called coordinate system in plane (space).

Traditionally, the axes in plane are called the axis of abscissas (axis O*x*) and the axis of ordinates (axis O*y*) and directed in the way that the shortest turn from positive semi-axis O*x* to positive semi-axis O*y* is made anticlockwise.

The axes in space are called the axis of abscissas (axis O*x*), the axis of ordinates (axis O*y*) and the applicate axis (axis O*z*) and directed in the way that the shortest turn from positive semi-axis O*x* to positive semi-axis O*y* is made anticlockwise if you observe this turn from the positive semi-axis O*z*.

Natural bases in plane and in space are bases formed from the unit vectors directed along the positive semi-axes.

Namely, natural basis in plane is the set of vectors

$$\vec{i}(1,0),\ \vec{j}(0,1);$$

natural basis in space is the set of vectors

$$\vec{i}(1,0,0),\ \vec{j}(0,1,0), \vec{k}(0,0,1).$$

From Note 2 to the Theorem about vector decomposition (Section 2.1.4) it follows that to find coordinates of the vector in the mentioned above bases we should connect the origin of the vector with point *O* and drop perpendiculars on the exes to find the vector-projections of this vector on basis vectors. In this case the vector is equal to sum of obtained vector-projections (Fig.13). Thus, we have:

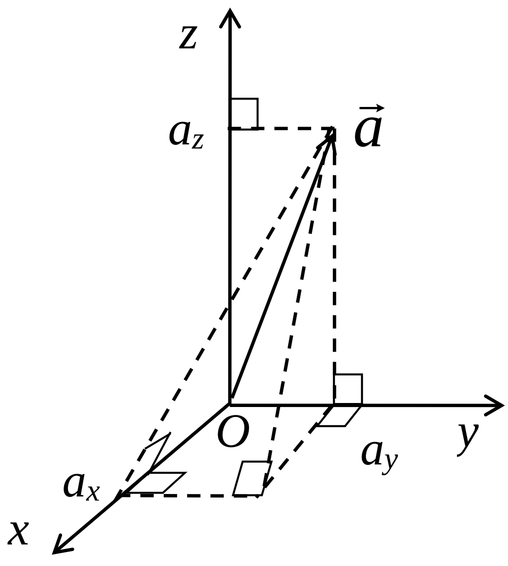


Figure13

in plane *Oxy*

$$\vec{a} = a_x\vec{i} + a_y\vec{j} = (a_x, a_y);$$

in space *Oxyz*

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = (a_x, a_y, a_z),$$

where

$$a_x = pr_{\vec{i}}\vec{a} = |\vec{a}|\cos\alpha, \quad a_y = pr_{\vec{j}}\vec{a} = |\vec{a}|\cos\beta, \quad a_z = pr_{\vec{k}}\vec{a} = |\vec{a}|\cos\gamma,$$

$\alpha = (\widehat{\vec{a},\vec{i}})$, $\beta = (\widehat{\vec{a},\vec{j}})$, $\gamma = (\widehat{\vec{a},\vec{k}})$ are angles between the vector and positive semi-axes *Ox*,*Oy*,Oz.

$\cos\alpha, \cos\beta, \cos\gamma$ are called *the direction cosines* of the vector.

From Fig.13 it follows that

1) By Pythagorean Theorem

$$\boxed{|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}}$$

2) $1 = \dfrac{|\vec{a}|^2}{|\vec{a}|^2} = \dfrac{|\vec{a}|^2 \cos^2\alpha + |\vec{a}|^2 \cos^2\beta + |\vec{a}|^2 \cos^2\gamma}{|\vec{a}|^2} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$, i.e.

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

3) Vector $(\cos\alpha, \cos\beta, \cos\gamma)$ is a vector of unit length with the same with vector $\vec{a}$ direction. Thus, this vector is ort of the vector $\vec{a}$, i.e.

$$\boxed{\vec{a}^0 = (\cos\alpha, \cos\beta, \cos\gamma) = \frac{\vec{a}}{|\vec{a}|}}$$

Example. It is known that $|\vec{a}| = 2$, $\cos\alpha = 1/2$, $\cos\gamma = -1/2$ and an angle between the axis O*y* and $\vec{a}$ is acute. Find the coordinates of the vector $\vec{a}$.

Since the angle β is acute then $\cos\beta > 0$ and

$$\cos\beta = \sqrt{1 - \cos^2\alpha - \cos^2\gamma} = \sqrt{1 - \frac{1}{4} - \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore

$a_x = |\vec{a}|\cos\alpha = 1$, $a_y = |\vec{a}|\cos\beta = \sqrt{2}$, $a_z = \vec{a}|\cos\gamma = -1$;

$\vec{a} = \left(1; \sqrt{2}; -1\right)$.

2.1.9. Radius-vector of the Point

Definition. Suppose we have Cartesian coordinate system. Vector $\overrightarrow{OM}$ with origin in the point *O* and a terminus *M* is called a radius-vector of the point *M*.

Coordinates of the point in the Cartesian coordinate system by definition are coordinates of its radius-vector, i.e.

$$\text{If} \quad \overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) \quad \text{then} \quad M(x, y, z).$$

Let us find coordinates of the vector $\overrightarrow{AB}$ through the coordinates of A and B (Fig.14).

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B, y_B, z_B) - (x_A, y_A, z_A) = (x_B - x_A, y_B - y_A, z_B - z_A).$$

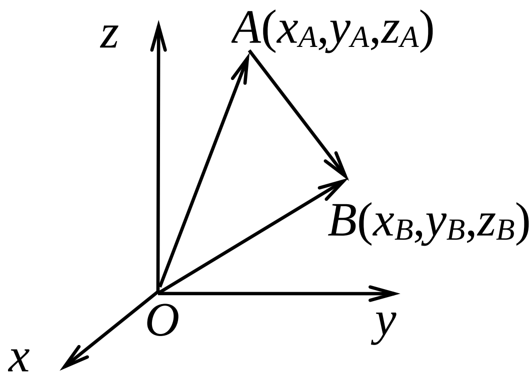


Figure 14

It means that to find coordinates of the vector we should subtract from the coordinates of the terminus the coordinates of the origin.

At the same time, since module of the vector $\overrightarrow{AB}$ is equal to the distance between two points, we state the following:

The distance between two points A and B is equal to

$$d = |\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

Example. It is known that $\vec{a} = \overrightarrow{AB} = (1, 2, -1)$, $A(1,1,0)$. Find the coordinates of the point B and distance between the points A and B.

$a_x = x_B - x_A \Rightarrow x_B = a_x + x_A = 1 + 1 = 2$;

$a_y = y_B - y_A \Rightarrow y_B = a_y + y_A = 2 + 1 = 3$;

$a_z = z_B - z_A \Rightarrow z_B = a_z + z_A = -1 + 0 = -1$.

Therefore, $B(2,3,-1)$.

The distance between the points A and B is equal to the length of the vector $\overrightarrow{AB}$:

$$d = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

2.1.10. Division of the Segment in the Given Ratio

Let us find coordinates of the point C which divides the segment AB in the ratio λ:μ, i.e. $|\overrightarrow{AC}|:|\overrightarrow{CB}|=\lambda:\mu$.

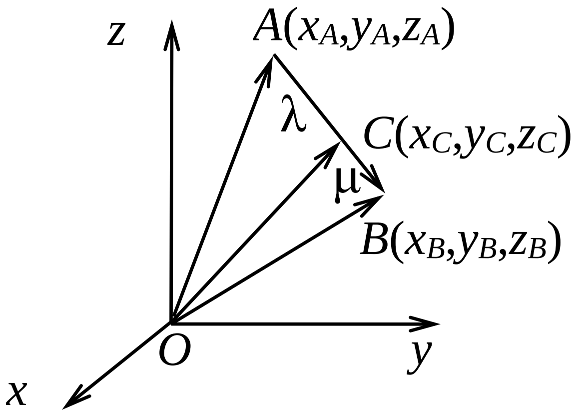


Figure 15

From Fig.15 it follows that

$$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OA}+\overrightarrow{AB}\frac{\lambda}{\lambda+\mu}=$$

$$=\overrightarrow{OA}+\left(\overrightarrow{OB}-\overrightarrow{OA}\right)\frac{\lambda}{\lambda+\mu}=\frac{\mu}{\lambda+\mu}\overrightarrow{OA}+\frac{\lambda}{\lambda+\mu}\overrightarrow{OB}=$$

$$=\left(\frac{\mu}{\lambda+\mu}x_A+\frac{\lambda}{\lambda+\mu}x_B,\ \frac{\mu}{\lambda+\mu}y_A+\frac{\lambda}{\lambda+\mu}y_B,\ \frac{\mu}{\lambda+\mu}z_A+\frac{\lambda}{\lambda+\mu}z_B\right)$$

Thus

$$C:\quad x_C=\frac{\mu x_A+\lambda x_B}{\lambda+\mu},\ y_C=\frac{\mu y_A+\lambda y_B}{\lambda+\mu},\ z_C=\frac{\mu z_A+\lambda z_B}{\lambda+\mu}.$$

Example 1. Let point M be a middle of the segment. In this case $\lambda=\mu=1$. Therefore

$$x_M=\frac{x_A+x_B}{2},\ y_M=\frac{y_A+y_B}{2},\ z_M=\frac{z_A+z_B}{2}.$$

Example 2. Find the point M of median intersection in the triangle with vertices $A(x_A,y_A,z_A)$, $B(x_B,y_B,z_B)$, $C(x_C,y_C,z_C)$ (Fig.16).

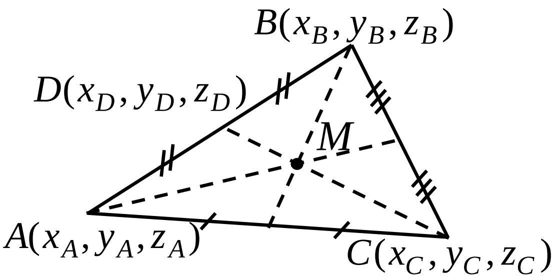


Figure 16

Since M is a point of median intersection, it divides each median in the ration 2:1. Therefore, the coordinates of this point could be found through the coordinates of the points C and D in the following way:

$$M: \quad x_M = \frac{x_C + 2x_D}{2+1}, \;\; y_M = \frac{y_C + 2y_D}{2+1}, \;\; z_M = \frac{z_C + 2z_D}{2+1},$$

where D is a middle of the side AB and therefore

$$x_D = \frac{x_A + x_B}{2}, \;\; y_D = \frac{y_A + y_B}{2}, \;\; z_D = \frac{z_A + z_B}{2}.$$

Thus,

$$M: \quad x_M = \frac{x_C + 2\dfrac{x_A + x_B}{2}}{2+1} = \frac{x_A + x_B + x_C}{3},$$

$$y_M = \frac{y_C + 2\dfrac{y_A + y_B}{2}}{2+1} = \frac{y_A + y_B + y_C}{3},$$

$$z_M = \frac{z_C + 2\dfrac{z_A + z_B}{2}}{2+1} = \frac{z_A + z_B + z_C}{3}.$$

Example 3. Find the center of the gravity of the triangle with vertices $A(1;2;3)$, $B(-1;3;4)$, $C(3,0,-2)$. Since the center of the gravity in triangle coincides with the point of median intersection, the coordinates of the center are:

$$x_o = \frac{x_A + x_B + x_C}{3} = \frac{1-1+3}{3} = 1, \;\; y_o = \frac{y_A + y_B + y_C}{3} = \frac{2+3+0}{3} = \frac{5}{3},$$

$$z_o = \frac{z_A + z_B + z_C}{3} = \frac{3+4-2}{3} = \frac{5}{3}.$$

Note. All obtained above formulas are valid for the points in plane, as well.

Except linear operations on vectors, such as addition and multiplication by scalar, there is an operation of vector multiplication. Moreover, It is possible to multiply vectors in three ways, namely in scalar, vector and mixed ways.

2.1.11. Scalar Product

Definition. Scalar product (or *dot product*) of two vectors $\vec{a}$ and $\vec{b}$ is a number (scalar) equal to $|\vec{a}||\vec{b}|\cos\alpha$, where α is an angle between vectors $\vec{a}$ and $\vec{b}$.

We denote the scalar product in two ways: $(\vec{a},\vec{b})$ or just $\vec{a}\vec{b}$ or $\vec{a}\cdot\vec{b}$. So,

$$(\vec{a},\vec{b})=|\vec{a}||\vec{b}|\cos\alpha.$$

Since

$$|\vec{b}|\cos\alpha = pr_{\vec{a}}\vec{b}, \qquad |\vec{a}|\cos\alpha = pr_{\vec{b}}\vec{a},$$

we have

$$(\vec{a},\vec{b})=|\vec{a}|pr_{\vec{a}}\vec{b}=|\vec{b}|pr_{\vec{b}}\vec{a},$$

$$pr_{\vec{b}}\vec{a}=\frac{(\vec{a},\vec{b})}{|\vec{b}|}.$$

Statement (Criterion of the perpendicularity) Two non-zero vectors are perpendicular if and only if their scalar product is equal to zero, i.e.

$$\bar{a}\perp\bar{b} \Leftrightarrow (\bar{a},\bar{b})=0.$$

Indeed,

$$\bar{a}\perp\bar{b} \Leftrightarrow \alpha=\frac{\pi}{2}\Leftrightarrow \cos\alpha=0\Leftrightarrow(\bar{a},\bar{b})=|\vec{a}||\vec{b}|\cos\alpha=0.$$

Algebraic properties of the scalar product:

1) $(\vec{a},\vec{b})=(\vec{b},\vec{a})$**;**

2) $(\lambda\vec{a},\vec{b})=(\vec{a},\lambda\vec{b})=\lambda(\vec{a},\vec{b})$;

3) $(\vec{a}+\vec{b},\vec{c})=(\vec{a},\vec{c})+(\vec{b},\vec{c})$.

Proof. 1) $(\vec{a},\vec{b})=|\vec{a}||\vec{b}|\cos\alpha=|\vec{b}||\vec{a}|\cos\alpha=(\vec{b},\vec{a})$;

2) $(\lambda\vec{a},\vec{b})=|\vec{b}|\,pr_{\vec{b}}\lambda\vec{a}=\lambda|\vec{b}|\,pr_{\vec{b}}\vec{a}=\lambda(\vec{a},\vec{b})$;

3) $(\vec{a}+\vec{b},\vec{c})=|\vec{c}|\;np_{\vec{c}}(\vec{a}+\vec{b})=|\vec{c}|(pr_{\vec{c}}\vec{a}+pr_{\vec{c}}\vec{b})=(\vec{a},\vec{c})+(\vec{b},\vec{c})$.

Properties are proven.

From the definition It follows that

$$(\vec{a},\vec{a})=|\vec{a}|^2 \quad \text{or} \quad |\vec{a}|=\sqrt{(\vec{a},\vec{a})}\,.$$

Thus, we obtain an additional fourth property of scalar product:

4) $(\vec{a},\vec{a})\geq 0$ and $(\vec{a},\vec{a})=0 \Leftrightarrow \vec{a}=\vec{0}$.

Example 1. It is known that $\vec{a}=5\vec{p}+2\vec{q}$, $\vec{b}=\vec{p}-3\vec{q}$, $|\vec{p}|=1$, $|\vec{q}|=2$, $\varphi=\left(\widehat{\vec{p},\vec{q}}\right)=\dfrac{\pi}{3}$. Find $|\bar{a}+\bar{b}|$.

By the last formula

$$|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b},\vec{a}+\vec{b})=(5\vec{p}+2\vec{q}+\vec{p}-3\vec{q},5\vec{p}+2\vec{q}+\vec{p}-3\vec{q})=(6\vec{p}-\vec{q},6\vec{p}-\vec{q})=$$

$$=36(\vec{p},\vec{p})-6(\vec{p},\vec{q})-6(\vec{q},\vec{p})+(\vec{q},\vec{q})=\text{[By properties of scalar product]}=$$

$$=36|\vec{p}|^2-12(\vec{p},\vec{q})+|\vec{q}|^2=36\cdot 1-12\cdot 1\cdot 2\cdot\cos\frac{\pi}{3}+2^2=36-12+4=28.$$

Thus,

$$|\bar{a}+\bar{b}|=\sqrt{28}=2\sqrt{7}\,\textbf{.}$$

Example 2. Find the ort of the vector.

From the definition of ort it follows that $\vec{a}^\circ=\lambda\vec{a}$, where $\lambda>0$. Therefore

$$1=(\vec{a}^\circ,\vec{a}^\circ)=(\lambda\vec{a},\lambda\vec{a})=\lambda^2|\vec{a}|^2\Rightarrow\lambda=\frac{1}{|\vec{a}|}\Rightarrow$$

$$\boxed{\vec{a}^{\circ} = \frac{\vec{a}}{|\vec{a}|}}$$

Note, that we have obtained the same formula as obtained above through the direction cosines.

Let us find the formula to calculate the scalar product of vectors given by their coordinates in the orthonormal basis $\vec{i}, \vec{j}, \vec{k}$.

Since

$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$$

and

$$\vec{i} \perp \vec{j}, \vec{i} \perp \vec{k}, \vec{j} \perp \vec{k}, \text{ i.e. } (\vec{i}, \vec{j}) = (\vec{i}, \vec{k}) = (\vec{j}, \vec{k}) = 0,$$

we have

$$(\vec{a}, \vec{b}) = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}, b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) = a_xb_x + a_yb_y + a_zb_z.$$

It means that to find the scalar product we should multiply the corresponding coordinates of vectors and then summarize these products.

Note. This formula is valid for the vectors in plane (case, when $a_z = b_z = 0$).

Example. It is known that $\vec{a}(1,2,3)$, $\vec{b}(-1,1,2)$, $\vec{c}(0,1,4)$. Find a value k such that $\vec{a} \perp \left(\vec{b} - k\vec{c}\right)$. By the criterion of the perpendicularity we have

$$0 = \left(\vec{a}, \vec{b} - k\vec{c}\right) = \left(\vec{a}, \vec{b}\right) - k\left(\vec{a}, \vec{c}\right) =$$
$$= 1(-1) + 2\cdot 1 + 3\cdot 2 - k(1\cdot 0 + 2\cdot 1 + 3\cdot 4) =$$
$$= 7 - 14k = 0.$$

Thus

$$k = \frac{7}{14} = \frac{1}{2}.$$

Note. The two and three-dimensional vector spaces with scalar product, satisfying four properties written above, are called *Euclidean vector spaces*.

2.1.12. Inner Product Space and Normed Space

An **inner product space** is a linear space of arbitrary (possibly infinite) dimension with additional structure, which, among other things, enables generalization of concepts from two or three-dimensional Euclidean geometry. The additional structure associates to each pair of elements in the space a number which is called the **inner product** (also called a **scalar product**) of the elements. Inner products allow the rigorous introduction of intuitive geometrical notions such as the angle between vectors or length of vectors in spaces of all dimensions. It also allows introduction of the concept of orthogonality between elements in the inner product space.

Definition. Suppose, the field of scalars denoted **F** is either the field of real numbers **R** or the field of complex numbers **C** (See the course of mathematical analysis). An inner product space is a vector space *V* over the field **F** together with a function called an inner (scalar) product such that it puts in correspondence to any two elements *x*, *y* of *V* the scalar from **F**, denoted $\langle x, y\rangle$ or (x, y), and It satisfies the following axioms

1) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$;

2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (line denotes the complex conjugate);

3) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$, $\lambda \in \mathbf{F}$;

4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

Definition. The inner product space over the field of real numbers is called the Euclidean space.

In the two or three-dimensional vectors spaces the idea of the "length" of a vector is intuitive and can be easily extended to any real vector space $\mathbf{R}^n$. It turns out that the following properties of "vector length" are the crucial ones:

1. The zero vector has zero length; every other vector has a positive length.
2. Multiplying a vector by a positive number changes its length without changing its direction.
3. The triangle inequality holds. That is, taking lengths as distances, the distance from *A* through *B* to *C* is never shorter than going directly from *A* to *C*, or the shortest distance between any two points is a straight line.

Their generalization for more abstract vector spaces, leads to the notion of *the norm*. A vector space on which a norm is defined is then called *a normed vector space*.

Definition. Linear space V is called a normed space if any element x of this space is associated with not-negative real number $\|x\|$ called norm of the element such that the following three axioms are valid:

1) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (nondegeneracy);

2) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbf{R}$ (homogeneity);

3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Note 1. In the inner product space the norm can be simply introduced by the formula

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Thus, any inner product space is a normed space as well.

Note 2. By means of the inner product and the norm the following concepts could be simply introduced:

– the length (or the norm) of the elements:

$$|x| = \|x\| = \sqrt{\langle x, x \rangle};$$

– the angle between two elements in Euclidian space:

$$\alpha = \arccos \frac{\langle x, y \rangle}{\|x\| \|y\|};$$

– the concept of orthogonality between elements:

$$x \perp y \Leftrightarrow \langle x, y \rangle = 0.$$

2.1.13. Vector Product

Definition. The ordered triple of uncomplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is called *a right–hand triple* if the shortest turn from the vector $\vec{a}$ to the vector $\vec{b}$ is made anticlockwise when their origins are connected and you observe this turn from the terminus of $\vec{c}$. In other case this triple is *a left – hand triple*.

Definition. Vector product (or *cross product*) of vectors $\vec{a}$ and $\vec{b}$ is a vector $\vec{c}$ satisfying the following three conditions:

1) $\vec{c} \perp \vec{a}$, $\vec{c} \perp \vec{b}$;

2) $|\vec{c}| = |\vec{a}||\vec{b}|\sin\alpha$, where α is an angle between $\vec{a}$ and $\vec{b}$;

3) $\vec{a}, \vec{b}, \vec{c}$ form the right–hand triple.

We denote vector product in two ways, namely $\vec{c} = \vec{a} \times \vec{b}$ or $\vec{c} = [\vec{a}, \vec{b}]$.

Algebraic properties of the vector product:

1) $[\vec{a}, \vec{b}] = -[\vec{b}, \vec{a}]$ (Property of anti-symmetry);

2) $[\lambda\vec{a}, \vec{b}] = \lambda[\vec{a}, \vec{b}] = [\vec{a}, \lambda\vec{b}]$;

3) $[\vec{a}, \vec{b} + \vec{c}] = [\vec{a}, \vec{b}] + [\vec{a}, \vec{c}]$.

Proof. Properties 1)-2) follow directly from conditions 2 and 3 of definition.

To prove the property 3) let us show first that there is another way to plot the result of vector product (Fig. 17). We connect the origins of two vectors, project

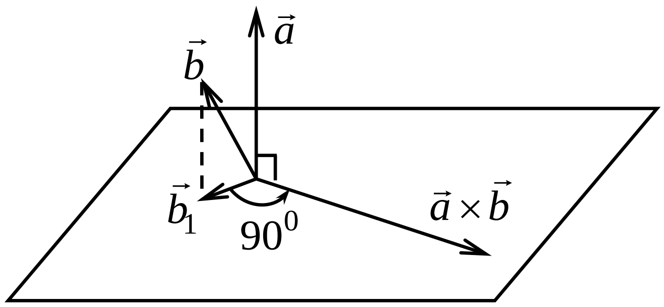


Figure 17

the vector $\vec{b}$ on the plane perpendicular to the vector $\vec{a}$. Then we turn the obtained vector $\vec{b}_1$ anticlockwise on 90 degrees and multiply by $|\vec{a}|$. The result is $\vec{a} \times \vec{b}$ since it satisfies all conditions from the definition.

We are going to use this procedure to prove the third property. Consider the parallelogram I from the Fig.18 and project it on the plane perpendicular to $\vec{a}$.

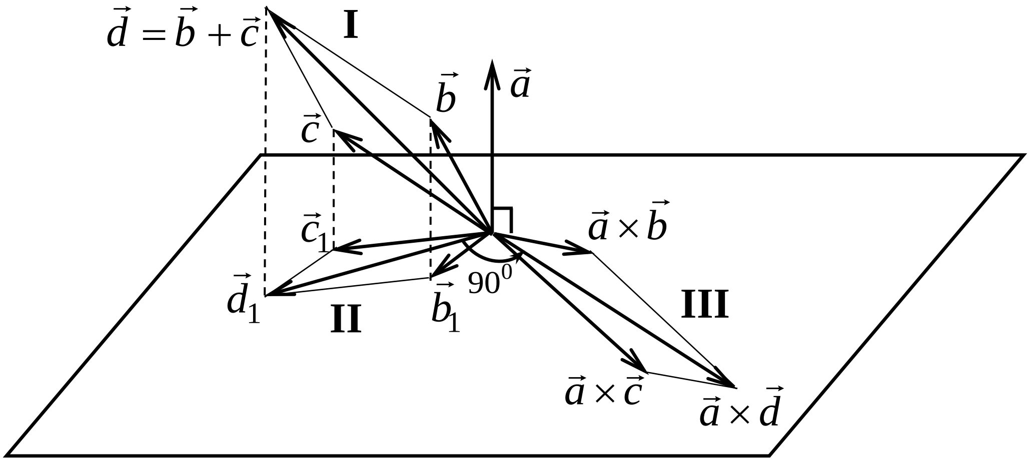


Figure 18

Obtained figure II is also parallelogram and, moreover, the diagonal $\vec{d}=\vec{b}+\vec{c}$ of the figure I is projected into the diagonal $\vec{d}_1=\vec{b}_1+\vec{c}_1$ of the figure II. To obtain the figure III we turn the figure II anticlockwise on 90 degrees and stretch it in $|\vec{a}|$ times. At that we again obtain the parallelogram where the diagonal of III is obtained by turn and stretching of the diagonal of II. It means that the obtained diagonal is the vector $\vec{a}\times\vec{d}=\vec{a}\times\left(\vec{b}+\vec{c}\right)$ equal to the sum of the parallelogram sides, i.e.

$$\vec{a}\times\vec{d}=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}\,.$$

Properties are proven.

Geometrical properties of the vector product:

1) $\vec{a}\parallel\vec{b}\Leftrightarrow\vec{a}\times\vec{b}=0$ (Criterion of collinearity of two non-zero vectors)

Indeed, $\vec{a}\parallel\vec{b}\Leftrightarrow\alpha=(\widehat{\vec{a},\vec{b}})=\left[\begin{array}{l}0\\ \pi\end{array}\right.\Leftrightarrow\sin\alpha=0\Rightarrow\left|\vec{a}\times\vec{b}\right|=0\Leftrightarrow\vec{a}\times\vec{b}=\vec{0}\,.$

Note. Another criterion of collinearity follows from definition, namely,

$$\vec{a}\parallel\vec{b}\Leftrightarrow\vec{a}=\lambda\vec{b}\Leftrightarrow a_x=\lambda b_x,a_y=\lambda b_y,a_z=\lambda b_z\Leftrightarrow\frac{a_x}{b_x}=\frac{a_y}{b_y}=\frac{a_z}{b_z},$$

i.e. *the coordinates of collinear vectors are proportional*.

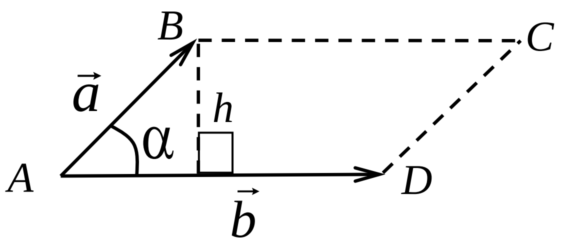


Figure 19

2) $S_{par} = |\vec{a} \times \vec{b}|$, i.e. the area of the parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$ is equal to the module of their vector product.

Indeed, from Fig. 19 we have

$$S_{par} = AB \cdot AD \cdot \sin\alpha = |\vec{a}||\vec{b}|\sin\left(\widehat{\vec{a},\vec{b}}\right) = |\vec{a} \times \vec{b}|.$$

3) The altitude of the parallelogram is equal to

$$h = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} .$$

Indeed, from Fig.19 It follows that:

$$S_{par} = h \cdot AD \Rightarrow h = \frac{S_{par}}{AD} = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} .$$

4) The area of the triangle, constructed on the vectors $\vec{a}$ and $\vec{b}$, is equal to a half of the module of their vector product. At the same time, the formula for the altitude dropped on the vector $\vec{b}$ is the same as for the parallelogram. So

$$S_{tr} = \frac{1}{2}|\vec{a} \times \vec{b}|, \qquad h = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} .$$

5) **Finding the vector perpendicular to the plane of any two uncollinear vectors.** Suppose, $\vec{a}$ and $\vec{b}$ are not collinear. Then some parallelogram which is planar figure can be constructed on them. Vector $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ and thus to the plane of the parallelogram. Therefore, any vector, perpendicular to the plane of two uncollinear vectors $\vec{a}$ and $\vec{b}$ is collinear to $\vec{a} \times \vec{b}$. So, we state for uncollinear non-zero vectors $\vec{a}$ and $\vec{b}$

$$\begin{cases} \vec{c} \perp \vec{a} \\ \vec{c} \perp \vec{b} \end{cases} \Rightarrow \vec{c} \parallel \vec{a} \times \vec{b} \Leftrightarrow \vec{c} = \lambda \vec{a} \times \vec{b},\ \lambda \in R \setminus \{0\}.$$

Let us find the formula to calculate the vector product of vectors given by their coordinates in the orthonormal basis $\vec{i}, \vec{j}, \vec{k}$. Since

$$\begin{array}{lll} \vec{i} \times \vec{i} = 0 & \vec{i} \times \vec{j} = \vec{k} & \vec{i} \times \vec{k} = -\vec{j} \\ \vec{j} \times \vec{i} = -\vec{k} & \vec{j} \times \vec{j} = 0 & \vec{j} \times \vec{k} = \vec{i} \\ \vec{k} \times \vec{i} = \vec{j} & \vec{k} \times \vec{j} = -\vec{i} & \vec{k} \times \vec{k} = 0 \end{array}$$

the vector product of vectors $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is equal to

$$\left[\vec{a}, \vec{b}\right] = \left[a_x\vec{i} + a_y\vec{j} + a_z\vec{k}, b_x\vec{i} + b_y\vec{j} + b_z\vec{k}\right] = a_xb_x\vec{i} \times \vec{i} + a_xb_y\vec{i} \times \vec{j} + a_xb_z\vec{i} \times \vec{k} + a_yb_x\vec{j} \times \vec{i} +$$

$$+ a_yb_y\vec{j} \times \vec{j} + a_yb_z\vec{j} \times \vec{k} + +a_zb_x\vec{k} \times \vec{i} + a_zb_y\vec{k} \times \vec{j} + a_zb_z\vec{k} \times \vec{k} =$$

$$= \left(a_xb_y - a_yb_x\right)\vec{i} \times \vec{j} + \left(-a_xb_z + a_zb_x\right)\vec{k} \times \vec{i} + \left(a_yb_z - a_zb_y\right)\vec{j} \times \vec{k} =$$

$$= \vec{i}\left(a_yb_z - a_zb_y\right) + (-1)\vec{j}\left(a_xb_z - a_zb_x\right) + \vec{k}\left(a_xb_y - a_yb_x\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

So,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

Example. Find area of the triangle with vertices in the points $A(1,1), B(2,-1), C(0,3)$ and vector $\vec{h}$ collinear to the altitude dropped on side *AB*.

Since the problem is formulated in plane we can not calculate vector product to find area. That is why before solving this problem we reformulate the task by expanding the coordinates of points to spatial case, i.e. we suppose that vertices have the following coordinates:

$$A(1,1,0), B(2,-1,0), C(0,2,0).$$

Then

$$\overrightarrow{AB} = (1,-2,0), \quad \overrightarrow{AC} = (-1,1,0),$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ -1 & 1 & 0\end{vmatrix}=0\vec{i}-0\vec{j}+(-1)\vec{k}=(0,0,-1),$$

$$S_{tr}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{0^2+0^2+(-1)^2}=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Vector $\vec{h}$ is perpendicular to the vector $\overrightarrow{AB}$ and to the vector $\overrightarrow{AB}\times\overrightarrow{AC}$ (since this vector is perpendicular to any vector in the plane of triangle). It means that

$$\vec{h}=\left[\overrightarrow{AB},\overrightarrow{AB}\times\overrightarrow{AC}\right]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ 0 & 0 & -1\end{vmatrix}=2\vec{i}+\vec{j}+0\vec{k}=(2,1,0).$$

These coordinates are coordinates in space. To get final answer we should save only the first two coordinates, i.e. $\vec{h}(2,1)$.

2.1.14. Mixed Product

Definition. Mixed product of vectors $\vec{a},\vec{b},\vec{c}$ is equal to the value obtained after scalar multiplication of the vector $\vec{c}$ by the vector product of vectors $\vec{a}$ and $\vec{b}$, i.e.

$$\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b},\vec{c}\right).$$

Theorem (Criterion of complanarity of three non-zero vectors) $\left(\vec{a},\vec{b},\vec{c}\right)=0\Leftrightarrow\vec{a},\vec{b},\vec{c}$ are complanar.

Proof. $\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b},\vec{c}\right)=0\Leftrightarrow\left[\begin{array}{l}\vec{a}\times\vec{b}\perp\vec{c}\\ \vec{a}\times\vec{b}=0\end{array}\right.$. It means that either $\vec{c}$ is parallel to the plane of $\vec{a}$ and $\vec{b}$ or $\vec{a}$ and $\vec{b}$ are collinear. In all these cases the vectors $\vec{a},\vec{b},\vec{c}$ are complanar. ***Theorem is proven.***

Note. If at least two factors coincide in the mixed product, this product is equal to zero. That is $(\vec{a},\vec{a},\vec{b})=0$.

Theorem (Mixed product of the right-hand triple) $\vec{a},\vec{b},\vec{c}$ form the right-hand triple if and only if $\left(\bar{a},\bar{b},\bar{c}\right)>0$.

Proof. From Fig.20 it follows that if $\vec{a},\vec{b},\vec{c}$ form the right-hand triple then an angle α is acute. Thus

$$\left(\bar{a}\times\bar{b},\bar{c}\right)=\left(\bar{a},\bar{b},\bar{c}\right)>0.$$

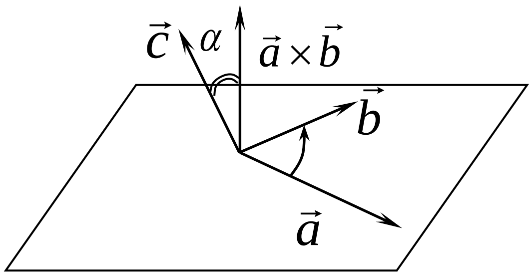


Figure 20

From the other hand, if $\left(\bar{a}\times\bar{b},\bar{c}\right)>0\Rightarrow\cos\alpha>0\Rightarrow$

$\Rightarrow\alpha$ is acute $\Rightarrow\bar{a},\bar{b},\bar{c}$ form the right-hand triple. ***Theorem is proven.***

Corollary.

$\left(\bar{a},\bar{b},\bar{c}\right)<0\Leftrightarrow\vec{a},\bar{b},\vec{c}$ form the left-hand triple.

Theorem (Geometrical meaning of the mixed product) $V_{parallelepiped}=\left|\left(\bar{a},\bar{b},\vec{c}\right)\right|$, i.e. the volume of the parallelepiped, constructed on the vectors $\vec{a},\vec{b},\vec{c}$, is equal to the module of their mixed product.

Proof. Suppose $\vec{a},\vec{b},\vec{c}$ is a right-hand triple (Fig.21). Then

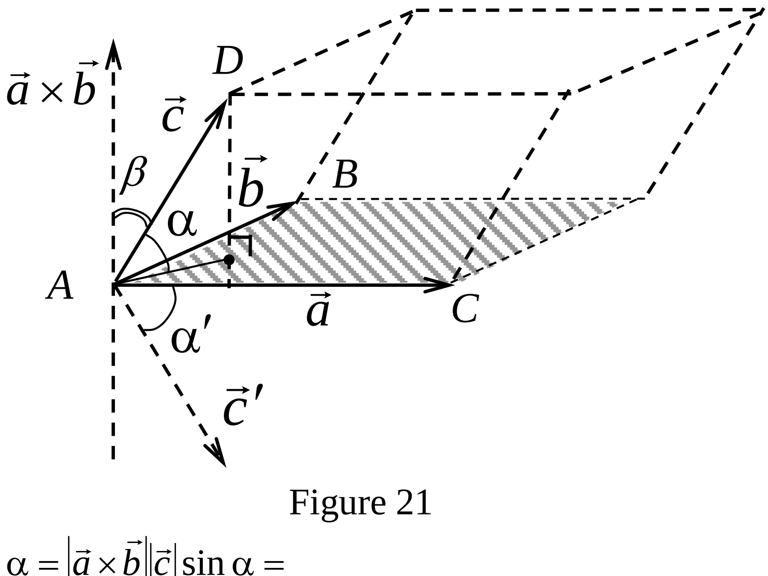


Figure 21

$$V=S\cdot AD\sin\alpha=\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\sin\alpha=$$

$$=\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\sin\left(\frac{\pi}{2}-\beta\right)=\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\cos\beta=\left(\vec{a}\times\vec{b},\vec{c}\right)=\left(\vec{a},\vec{b},\vec{c}\right)=\left|\left(\bar{a},\bar{b},\bar{c}\right)\right|.$$

If $\vec{a},\vec{b},\vec{c}$ form the left-hand triple (for this case $\vec{c}$ and α are shown as $\vec{c}',\alpha'$ on Fig.21) then

$$\sin\alpha = \sin\left(\beta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \beta\right) = -\cos\beta .$$

Therefore $V = -\left(\vec{a},\vec{b},\vec{c}\right) = \left|\left(\vec{a},\vec{b},\vec{c}\right)\right|$. ***Theorem is proven.***

Note. It is simple to check that if $\vec{a},\vec{b},\vec{c}$ is a right-hand triple then $\vec{c},\vec{a},\vec{b}$ and $\vec{b},\vec{c},\vec{a}$ form the right-hand triples, as well. Hence,

$$V = (\vec{a},\vec{b},\vec{c}) = (\vec{c},\vec{a},\vec{b}) = (\vec{b},\vec{c},\vec{a}) .$$

In the same way it can be shown that

$$V = -(\vec{b},\vec{a},\vec{c}) = -(\vec{c},\vec{b},\vec{a}) = -(\vec{a},\vec{c},\vec{b}) .$$

Moreover, from the obtained above it follows that

$$(\vec{a},\vec{b},\vec{c}) = (\vec{a}\times\vec{b},\vec{c}) = (\vec{b},\vec{c},\vec{a}) = (\vec{b}\times\vec{c},\vec{a}) = (\vec{a},\vec{b}\times\vec{c}),$$

i.e. to find mixed product we can multiply any two neighbour vectors in the vector way and then multiply the result vector by the third one in the scalar way.

Algebraic properties of the mixed product:

1) *a*) $\left(\vec{a},\vec{b},\vec{c}\right) = \left(\vec{c},\vec{a},\vec{b}\right) = \left(b,\vec{c},\vec{a}\right)$,

b) $\left(\vec{a},\vec{b},\vec{c}\right) = -\left(\vec{b},\vec{a},\vec{c}\right) = -\left(\vec{a},\vec{c},\vec{b}\right) = -\left(\vec{c},\vec{b},\vec{a}\right)$;

i.e. cyclic transposition of vectors does not change the value of the mixed product, but the transposition of any two neighbour vectors changes the sign of the mixed product. It follows from the last Note or from the properties of scalar and vector products.

2) $\left(\lambda\vec{a},\vec{b},\vec{c}\right) = \lambda\left(\vec{a},\vec{b},\vec{c}\right) = \left(\vec{a},\lambda\vec{b},\vec{c}\right) = \left(\vec{a},\vec{b},\lambda\vec{c}\right)$;

3) $\left(\vec{a}+\vec{b},\vec{c},\vec{d}\right) = \left(\vec{a},\vec{c},\vec{d}\right) + \left(\vec{b},\vec{c},\vec{d}\right)$.

Last two properties follow directly from the properties of scalar and vector products.

Geometrical properties of the mixed product:

1) $V_{parallelepiped} = \left|\left(\vec{a},\vec{b},\vec{c}\right)\right|$ (Fig.22)

2) The altitude of the parallelepiped dropped on the base of vectors $\vec{a}$ and $\vec{b}$ is

$$h = \frac{V}{S} = \frac{\left|\left(\vec{a},\vec{b},\vec{c}\right)\right|}{\left|\vec{a}\times\vec{b}\right|}.$$

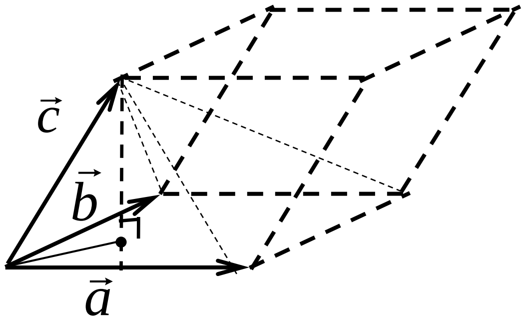


Figure 22

3) The volume of the tetrahedron constructed on vectors $\vec{a},\vec{b},\vec{c}$ (Fig.22) is equal to

$$V_{tetrahedron} = \frac{1}{6}V_{par.} = \frac{1}{6}\left|\left(\vec{a},\vec{b},\vec{c}\right)\right|.$$

The altitude of the tetrahedron coincides with the altitude of the parallelepiped, so it could be found by the same formula.

Let us find the formula to calculate the mixed product of vectors given by their coordinates in the orthonormal basis $\vec{i},\vec{j},\vec{k}$.

Suppose,

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = \left(a_x, a_y, a_z\right)$$

$$\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k} = \left(b_x, b_y, b_z\right)$$

$$\vec{c} = c_x\vec{i} + c_y\vec{j} + c_z\vec{k} = \left(c_x, c_y, c_z\right)$$

Let us evaluate $\left(\overline{a},\overline{b},\vec{c}\right) = \left(\overline{a},\overline{b}\times\overline{c}\right)$:

$$\overline{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \vec{i}\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \vec{j}\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \vec{k}\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} =$$

$$\left(\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix}, -\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix}, \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}\right).$$

$$\left(\vec{a},\vec{b}\times\vec{c}\right) = a_x\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Therefore,

$$\left(\vec{a},\vec{b},\vec{c}\right)=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Example 1. Find the coordinates of the vertex *D* of the tetrahedron *ABCD* if the volume of this tetrahedron is equal to 10, *D* is situated on the positive semi-axis Oz and $A(1;2;3)$, $B(-1;0;4)$, $C(0,4,1)$.

From condition it follows that *D* has coordinates $D(0;0;z_D)$ and

$$V=10=\frac{1}{6}\left|\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)\right|, \text{ i.e. } \left|\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)\right|=60.$$

But

$$\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)=\begin{vmatrix} -2 & -2 & 1 \\ -1 & 2 & -2 \\ -1 & -2 & z_D-3 \end{vmatrix}=-4(z_D-3)-4+2+2+8-2(z_D-3)=$$

$=-6z_D+26$.

Therefore

$$-6z_D+26=\pm 60 \Leftrightarrow \left[\begin{array}{l} -6z_D=34 \\ -6z_D=-86 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} z_D=-34/6=-17/3 \\ z_D=43/3 \end{array}\right.$$

Since *D* is situated on the positive semi-axis Oz the answer is $D(0;0;43/3)$.

Example 2. Prove that four points are situated on the same plane if their coordinates are $A(1;1;1)$, $B(1;2;3)$, $C(2;3;4)$, $D(0;2;4)$.

These points are from the same plane if and only if the vectors $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are complanar. Let us check this statement.

$$\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)=\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{vmatrix}=0-3+2+4-3-0=0.$$

Therefore the vectors are complanar and points are situated on the same plane.

Example 3. Find $\left(\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}\right)$ if $\left(\vec{a},\vec{b},\vec{c}\right)=1$. By the mixed product properties:

$$(\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c})=(\vec{a},\vec{b},\vec{c})+(\vec{a},\vec{c},\vec{c})+(\vec{b},\vec{b},\vec{c})+(\vec{b},\vec{c},\vec{c})=1+0+0+0=1.$$

2.2. Surfaces and Lines in Space R^3

Definition. Surface in the three-dimensional linear space R^3 is a locus of points in space with coordinates in Cartesian coordinate system satisfying the following equation

$$F(x, y, z) = 0.$$

The last equation is called the equation of the surface if the coordinates of all points of this surface, and only of those points, satisfy the equation.

Example 1. Suppose $F(x, y, z) = x^2 + y^2 + (z-1)^2 - 9 = 0$. Then $x^2 + y^2 + (z-1)^2 = 9$ or $\sqrt{x^2 + y^2 + (z-1)^2} = 3$. That is a locus of points with distance from point $(0,0,1)$ equal to 3, i.e. this equation is the equation of the sphere with center in the point $(0,0,1)$ and radius equal to 3 (Fig.23a).

Example 2. Suppose $F(x, y, z) = x^2 + y^2 - 9 = 0$. Then $x^2 + y^2 = 9$ or $\sqrt{x^2 + y^2} = 3$. That is a locus of points with distance from the axis Oz equal to

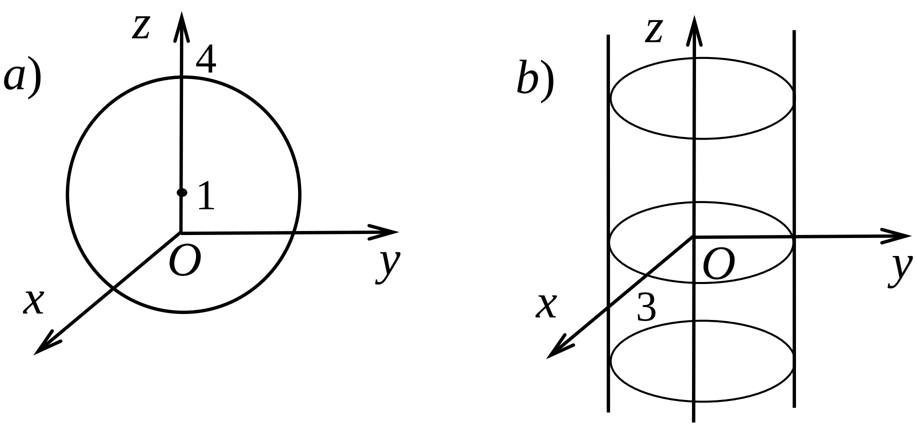


Figure 23

3, i.e. this equation is the equation of cylinder (Fig.23b).

Definition. Line in the three-dimensional linear space R^3 is a locus of points situated in the intersection of two surfaces.

The general equation of the line in R^3 looks like

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

Example 1. $\begin{cases} x^2 + y^2 = 9 \\ z = 2 \end{cases}$ That is a locus of points in the intersection of the cylinder from previous example and plane parallel to *Oxy* , i.e. that is a circle (Fig.24*a*).

Example 2. $\begin{cases} y^2 = x \\ z = 2 \end{cases}$ That is a locus of points in the intersection of the cylinder with directional line (directrix) $y^2 = x$ and the generatrix parallel to *Oz* and plane parallel to *Oxy*, i.e. this equation is the equation of parabola in plane $z = 2$ (Fig.24*b*).

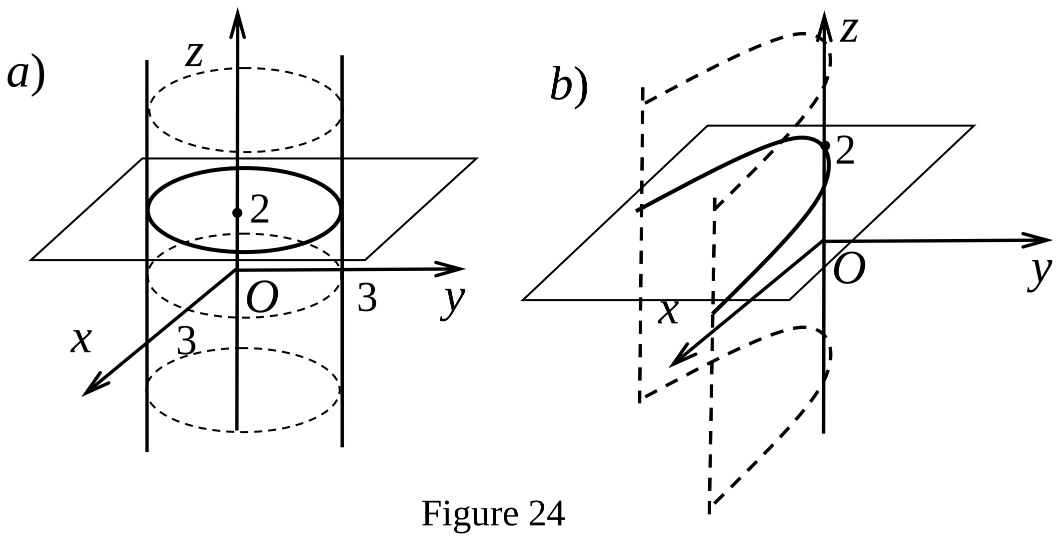


Figure 24

Note. Sometimes it is possible to express from the general equation of line two variables through the third one or all three variables through other value called a parameter. The obtained equations are called parametric equations of the line.

Example 1. $\begin{cases} y^2 = x \\ z = 2 \end{cases}$ Here *y* can be chosen as parameter. Then we have the following parametric equations of the same line:

$$\begin{cases} x = x(t) = t^2 \\ y = y(t) = t \qquad t \in R \\ z = z(t) = 2 \end{cases}$$

Example 2. The general equation of the line $\begin{cases} x^2 + y^2 = 9 \\ z = 2 \end{cases}$ can be written as

$$\begin{cases} x = x(t) = 3\cos t \\ y = y(t) = 3\sin t \qquad t \in [0, 2\pi], \\ z = z(t) = 2 \end{cases}$$

where t is an angle in plane Oxy between the positive semi-axis Ox and vector (x,y) (Fig.25).

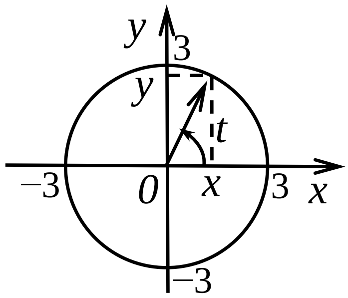


Figure 25

2.2.1. General Equation of Plane in Space

Theorem (about general equation of plane) Suppose x, y, z are the coordinates of a point in the Cartesian coordinate system. Any linear equation $Ax + By + Cz + D = 0$, where $A^2 + B^2 + C^2 \neq 0$, is an equation of plane in space.

Proof. Suppose coordinates of point (x_0, y_0, z_0) satisfy the equation $Ax + By + Cz + D = 0$, and denote the coordinates of any other point satisfying this equation by (x, y, z). then

$$Ax + By + Cz + D = 0, \tag{*}$$

$$Ax_0 + By_0 + Cz_0 + D = 0. \tag{**}$$

After subtraction of equation (*) from equation (**) we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{***}$$

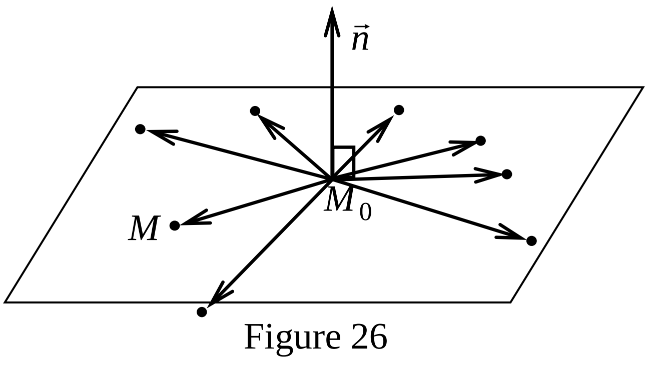


Figure 26

Equation (***) can be considered as zero scalar product of vector $\vec{n}(A, B, C)$ and vector $\overrightarrow{M_0M}(x - x_0, y - y_0, z - z_0)$. It means that for any point $M(x,y,z)$ with coordinates satisfying the equation

(*) the vector $\overline{M_0M} \perp \vec{n}$, i.e. all points satisfying this linear equation belong to the plane perpendicular to the vector $\vec{n}(A,B,C)$ (Fig.26). Moreover, from (**) we have that $D = -Ax_0 - By_0 - Cz_0$, where (x_0, y_0, z_0) satisfies (*).

From the other side the opposite statement is valid as well, i.e. any point $M(x,y,z)$ of the plane satisfies the equation (*). Indeed, two points of this plane M and M_0 form vector in plane perpendicular to the vector $\vec{n}(A,B,C)$. So,

$$0 = A(x - x_0) + B(y - y_0) + C(z - z_0) = Ax + By + Cz - Ax_0 - By_0 - Cz_0 =$$
$$= Ax + By + Cz + D,$$

where $D = -Ax_0 - By_0 - Cz_0$. ***Theorem is proven.***

Definition. Vector $\vec{n}(A,B,C)$ is called the normal vector of plane.

Vector $\vec{n}$ gives *an orientation* of the plane.

To describe some certain plane we have to determine also *a location* which can be given by any point of this plane (Fig.27).

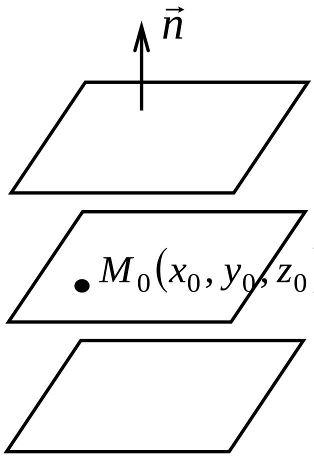


Figure 27

Definition. Equation

$$Ax + By + Cz + D = 0$$

is called *the general equation of the plane*.

Definition. Equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

is called the equation of the plane with given normal vector $\vec{n}(A,B,C)$ and a point of the plane $M_0(x_0, y_0, z_0)$.

Example. Let us find an equation of the plane perpendicular to the axis Oz and passing through the point $M_0(1;-2;3)$. Since this plane is perpendicular to the axis Oz It is perpendicular to the vector $\vec{k}(0,0,1)$ and this vector can be chosen as a normal vector of the plane. Therefore, $\vec{n}(A,B,C) = (0,0,1)$, $M_0(x_0, y_0, z_0) = (1;-2;3)$ and the equation of this plane looks like

$$0(x-1) + 0(y-(-2)) + 1(z-3) = 0 \Leftrightarrow z - 3 = 0 \Leftrightarrow z = 3.$$

2.2.2. Equation of Plane with Given Intercepts

Suppose $A \cdot B \cdot C \cdot D \neq 0$. Let us divide the general equation of the plane by $-D$. Then

$$\frac{Ax}{-D}+\frac{By}{-D}+\frac{Cz}{-D}=1$$

or

$$\frac{x}{-\frac{D}{A}}+\frac{y}{-\frac{D}{B}}+\frac{z}{-\frac{D}{C}}=1$$

or

$$\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1},$$

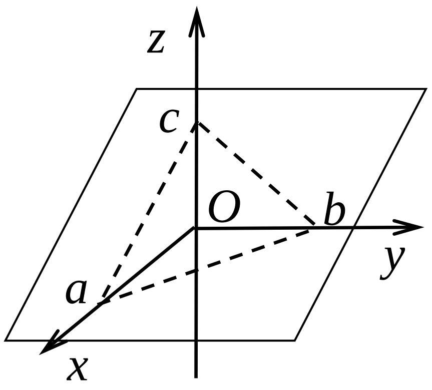


Figure 28

where $a=-\dfrac{D}{A}$, $b=-\dfrac{D}{B}$, $c=-\dfrac{D}{C}$ are the segments cut from the semi-axes of axes Ox,Oy,Oz or the intercepts (Fig.28).

The last equation is called the equation of plane with the given intercepts.

Example. Let us find an equation of the plane with equal intercepts and passing through the point $M_0(1;-2;3)$. Since the intercepts are equal the equation has a form:

$$\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1.$$

Since the point $M_0(1;-2;3)$ belongs to this plane the coordinates of this point satisfy the equation of the plane and therefore

$$\frac{1}{a}+\frac{-2}{a}+\frac{3}{a}=1 \Leftrightarrow \frac{2}{a}=1 \Leftrightarrow a=2.$$

Finally we obtain the equation

$$\frac{x}{2}+\frac{y}{2}+\frac{z}{2}=1 \quad \text{or} \quad x+y+z-2=0.$$

2.2.3. Angle Between Two Planes. Parallel and Perpendicular Planes

Definition. An angle between two planes is the angle between their normal vectors (Fig.29).

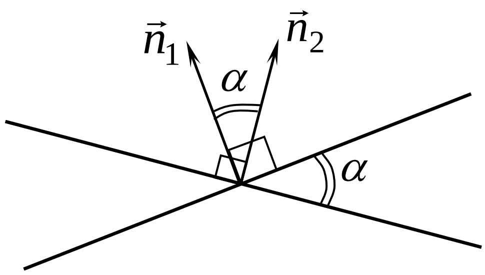


Figure 29

From definition we have:

$$\cos\alpha = \frac{(\bar{n}_1, \bar{n}_2)}{|\bar{n}_1||\bar{n}_2|} \Leftrightarrow \alpha = \arccos\frac{(\bar{n}_1, \bar{n}_2)}{|\bar{n}_1||\bar{n}_2|},$$

where $\bar{n}_1(A_1, B_1, C_1)$ and $\bar{n}_2(A_2, B_2, C_2)$ are the normal vectors of the planes

$$\text{plane 1: } A_1x + B_1y + C_1z + D_1 = 0,$$

$$\text{plane 2: } A_2x + B_2y + C_2z + D_2 = 0.$$

Therefore, conditions of parallel and perpendicular planes look like:

$$\text{Plane 1} \parallel \text{Plane 2} \Leftrightarrow \bar{n}_1 \| \bar{n}_2 \Leftrightarrow \bar{n}_1 \times \bar{n}_2 = 0 \Leftrightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2};$$

$$\text{Plane 1} \perp \text{Plane 2} \Leftrightarrow \bar{n}_1 \perp \bar{n}_2 \Leftrightarrow (\bar{n}_1, \bar{n}_2) = 0 \Leftrightarrow A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

Example 1. Find the value α such that the following planes are perpendicular: $\alpha x + y - 3z + 1 = 0$, $x + 5z - 19 = 0$. Since these planes are perpendicular then the scalar product of their normal vectors $\vec{n}_1(\alpha, 1, -3)$ and $\vec{n}_2(1,0,5)$ is equal to zero and we have

$$(\bar{n}_1, \bar{n}_2) = 0 = \alpha + 0 - 15 = \alpha - 15 \Leftrightarrow \alpha = 15.$$

Example 2. Find the values α and β such that two planes $\alpha x + y - 3z + 1 = 0$, $x - y + \beta z - 19 = 0$ are parallel. Since these planes are parallel then the coordinates of their normal vectors $\vec{n}_1(\alpha, 1, -3)$ and $\vec{n}_2(1, -1, \beta)$ are proportional and we have

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \Leftrightarrow \frac{\alpha}{1} = \frac{1}{-1} = \frac{-3}{\beta} \Leftrightarrow \begin{cases} \dfrac{\alpha}{1} = -1 \\ \dfrac{-3}{\beta} = -1 \end{cases} \Leftrightarrow \begin{cases} \alpha = -1 \\ \beta = 3 \end{cases}$$

Example 3. Find the angle between the planes $x + y - 3z + 1 = 0$, $x - y + z - 19 = 0$. Here $\vec{n}_1(1,1,-3)$ and $\vec{n}_2(1,-1,1)$. Thus

$$\alpha = \arccos\frac{(\bar{n}_1, \bar{n}_2)}{|\bar{n}_1||\bar{n}_2|} = \arccos\frac{1-1-3}{\sqrt{1^2+1^2+(-3)^2}\sqrt{1^2+(-1)^2+1^2}} = \arccos\frac{-3}{\sqrt{11}\sqrt{3}} =$$

$$= \arccos\left(-\sqrt{\frac{3}{11}}\right) = \pi - \arccos\sqrt{\frac{3}{11}}.$$

2.2.4. Distance from Point to Plane

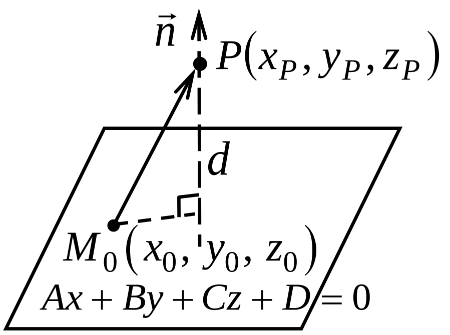


Figure 30

Let us find the distance from the point $P(x_P, y_P, z_P)$ to the plane $Ax + By + Cz + D = 0$.

Suppose $M_0(x_0, y_0, z_0)$ belongs to this plane. Then

$$Ax_0 + By_0 + Cz_0 + D = 0 \text{ or}$$

$$D = -Ax_0 - By_0 - Cz_0.$$

Distance from the point P to the plane can be found as (Fig.30)

$$d = \left|pr_{\vec{n}}\overline{M_0P}\right| = \left|\frac{(\bar{n}, \overline{M_0P})}{|\bar{n}|}\right| =$$

$$= \frac{|A(x_P - x_0) + B(y_P - y_0) + C(z_P - z_0)|}{\sqrt{A^2+B^2+C^2}} =$$

$$= \frac{|Ax_P + By_P + Cz_P - Ax_0 - By_0 - Cz_0|}{\sqrt{A^2+B^2+C^2}} = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2+B^2+C^2}}.$$

Thus

$$d = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2+B^2+C^2}}.$$

Example 1. Suppose $x_P = y_P = z_P = 0$. Then the distance from the origin to the plane is equal to

$$d = d_0 = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example 2. Find the distance from the point $P(1;-2;3)$ to the plane $x + 2y - 2z + 5 = 0$. By formula we have

$$d = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 + 2(-2) - 2 \cdot 3 + 5|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{|-4|}{\sqrt{9}} = \frac{4}{3}.$$

2.2.5. Normal Equation of Plane

Let us consider the general equation of the plane $Ax + By + Cz + D = 0$.

After division of the plane equation by $\sqrt{A^2 + B^2 + C^2}$ and renaming the coefficients we obtain the following:

$$\underbrace{\frac{A}{\sqrt{A^2 + B^2 + C^2}}}_{\cos\alpha} x + \underbrace{\frac{B}{\sqrt{A^2 + B^2 + C^2}}}_{\cos\beta} y + \underbrace{\frac{C}{\sqrt{A^2 + B^2 + C^2}}}_{\cos\gamma} z + \underbrace{\frac{D}{\sqrt{A^2 + B^2 + C^2}}}_{p} = 0$$

or

$$\cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z + p = 0.$$

Here $\left(\frac{A}{\sqrt{A^2 + B^2 + C^2}}, \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \frac{C}{\sqrt{A^2 + B^2 + C^2}} \right) = (\cos\alpha, \cos\beta, \cos\gamma)$ is the ort of the normal vector $\vec{n}^0$;

$|p| = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}} = d_0$ is distance from the origin to the plane. So, p is a distance taken with the sign plus or minus depending on the sign of the coefficient D.

The obtained equation $\cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z + p = 0$ is called *the normal equation of the plane*.

Example. Find the equations of the planes with distance from the plane $x + y - 3z + 1 = 0$ equal to $3\sqrt{11}$. These planes are parallel and therefore they have the same normal vector $\vec{n}(1,1,-3)$. Let us consider the normal equations of these three planes.

$|\vec{n}| = \sqrt{1+1+9} = \sqrt{11}$;

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + \frac{1}{\sqrt{11}} = 0;$$

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + p_1 = 0;$$

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + p_2 = 0.$$

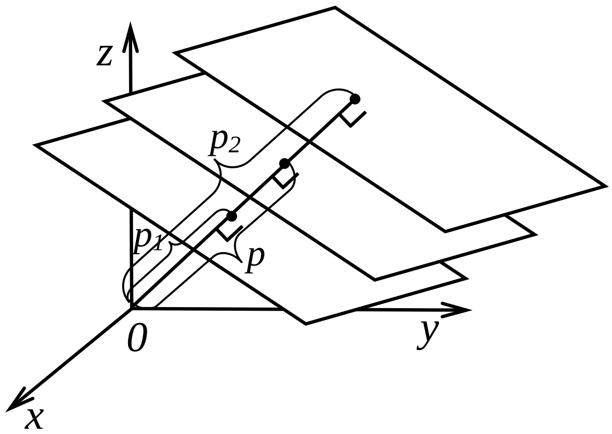


Figure 31

So, for the initial plane

$$p = \frac{1}{\sqrt{11}}.$$

From the condition and Fig.31 It follows that

$$p_1 = p - 3\sqrt{11} = \frac{1}{\sqrt{11}} - \frac{33}{\sqrt{11}} = -\frac{32}{\sqrt{11}},$$

$$p_2 = p + 3\sqrt{11} = \frac{1}{\sqrt{11}} + \frac{33}{\sqrt{11}} = \frac{34}{\sqrt{11}}$$

and therefore the asked equations are

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z - \frac{32}{\sqrt{11}} = 0 \Leftrightarrow x + y - 3z - 32 = 0;$$

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + \frac{34}{\sqrt{11}} = 0 \Leftrightarrow x + y - 3z + 34.$$

2.2.6. Three Particular Cases for Plane Equations

Case 1. Suppose we know one point $M_0(x_0, y_0, z_0)$ of the plane and any two uncollinear vectors $\vec{a}, \vec{b}$ parallel to this plane (Fig.32*a*).

In this case we have a point and to get equation of the plane we should just find the normal vector. But

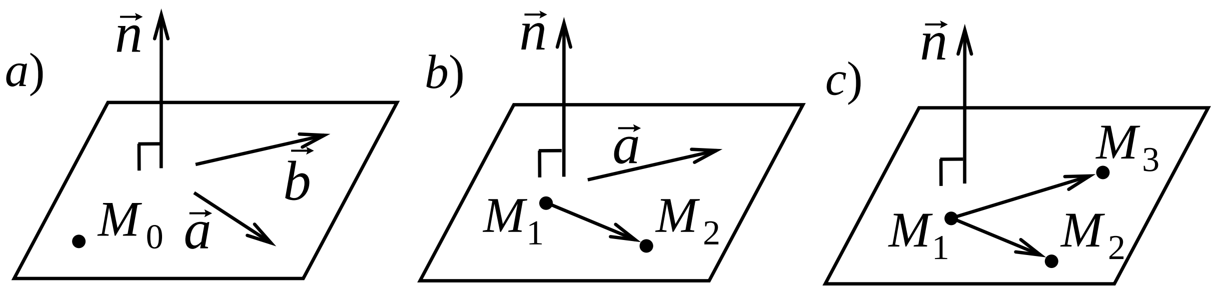


Figure 32

$$\begin{cases} \vec{n} \perp \vec{a} \\ \vec{n} \perp \vec{b} \end{cases} \Rightarrow \vec{n} \parallel \vec{a} \times \vec{b}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \vec{a} \times \vec{b},$$

where $\lambda \in R$, $\lambda \neq 0$.

Case 2. Suppose we know two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ of the plane and a vector $\vec{a}$ parallel to this plane and uncollinear to the vector $\overrightarrow{M_1M_2}$ (Fig.32*b*).

In this case we have a point but do not have a normal vector. Since

$$\begin{cases} \vec{n} \perp \vec{a} \\ \vec{n} \perp \overrightarrow{M_1M_2} \end{cases} \Rightarrow \vec{n} \parallel \vec{a} \times \overrightarrow{M_1M_2}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \vec{a} \times \overrightarrow{M_1M_2},$$

where $\lambda \in R$, $\lambda \neq 0$.

Case 3. Suppose we know three points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_2(x_2, y_2, z_2)$ of the plane such that vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_1M_3}$ are uncollinear (Fig.32*c*).

Since

$$\begin{cases} \vec{n} \perp \overrightarrow{M_1M_2} \\ \vec{n} \perp \overrightarrow{M_1M_3} \end{cases} \Rightarrow \vec{n} \parallel \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3},$$

where $\lambda \in R$, $\lambda \neq 0$. Then

$$\vec{n} = \lambda \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3} = \lambda \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = (A, B, C).$$

Thus, the equation of the plane passing through three given points has the form

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \text{ or}$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

Note. The last equation of the plane can be obtained directly from the condition of complanarity for vectors $\overrightarrow{M_1M_2}$, $\overrightarrow{M_1M_3}$, $\overrightarrow{M_1M}$, where point $M(x,y,z)$ is an arbitrary point of this plane.

Example. Find equation of the plane passing through the points $M_1(1,2,-1)$, $M_2(0,3,0)$ and $M_2(2,-1,1)$.

From the last formula we have

$$\begin{vmatrix} x-1 & y-2 & z+1 \\ 0-1 & 3-2 & 0+1 \\ 2-1 & -1-2 & 1+1 \end{vmatrix} = \begin{vmatrix} x-1 & y-2 & z+1 \\ -1 & 1 & 1 \\ 1 & -3 & 2 \end{vmatrix} =$$

$$= 5(x-1) + 3(y-2) + 2(z+1) = 5x + 3y + 2z - 9 = 0.$$

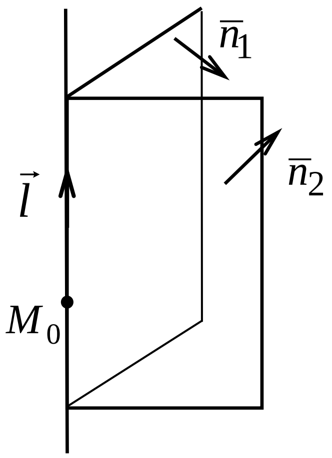


Figure 33

2.2.7. General Equation of Straight Line in Space

If two planes are not parallel then they intersect and a line of their intersection is called the straight line (Fig.33).

From definition it follows that all points of straight line satisfy the following system of equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

This system is called *the general equation of the straight line*.

Definition. Vector $\vec{l}$ parallel to the straight line is called the direction (or directing) vector of this straight line.

Vector $\vec{l}\,(m, n, p)$ determines *an orientation* of this straight line.

Since there are several straight lines with the same orientation, to describe straight line in the unique way we should determine any point $M_0(x_0, y_0, z_0)$ of this straight line (Fig.33). This point determines *a location* of this straight line.

There is a question: how to find the coordinates of point $M_0(x_0, y_0, z_0)$ and the direction vector $\vec{l}\,(m, n, p)$ from the general equation of the straight line?

Since two planes are not parallel their normal vectors are not collinear. Therefore

$$\begin{cases} \vec{l} \perp \vec{n}_1 \\ \vec{l} \perp \vec{n}_2 \end{cases} \Rightarrow \vec{l} = \lambda \vec{n}_1 \times \vec{n}_2 .$$

Moreover, at least one of equalities $\frac{A_1}{A_2} = \frac{B_1}{B_2}$, $\frac{A_1}{A_2} = \frac{C_1}{C_2}$, $\frac{B_1}{B_2} = \frac{C_1}{C_2}$ is false, that is at least one of numbers $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$ $\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}$ $\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}$ is not 0 and therefore rank of the matrix $\begin{pmatrix} A_1 & \mathrm{B}_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ is equal to 2 and the system $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$ has infinite number of solutions.

So we can evaluate coordinates of point M_0 by assigning to one of coordinates some constant value and evaluating other coordinates.

Example. Find the direction vector and one point of the straight line

$$\begin{cases} 2x - y + 2z = 1, \\ -x + y + 3z = 4. \end{cases}$$

Here

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ -1 & 1 & 3 \end{vmatrix} = -5\vec{i} - 8\vec{j} + \vec{k} = (-5,-8,1).$$

Then as direction vector we can take, for example, the vector $\vec{l} = -\vec{n}_1 \times \vec{n}_2 = (5,8,-1)$.

Let us write down the matrix of the system:

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Since $\begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} \neq 0$ then variable y is free.

$$y := 4: \quad \begin{cases} 2x + 2z = 5 \\ -x + 3z = 0 \end{cases} \Leftrightarrow \begin{matrix} x = 3z \\ 6z + 2z = 5 \end{matrix} \Leftrightarrow z = \frac{5}{8},\ x = \frac{15}{8}.$$

Thus we obtained the point $M_0\left(\frac{15}{8}, 4, \frac{5}{8}\right)$.

2.2.8. Canonical Equations of Straight Line

Suppose we know the coordinates of the point $M_0(x_0, y_0, z_0)$ and the direction vector $\vec{l}(m,n,p)$ of the straight line. Then for any point $M(x,y,z)$ of this straight line we have (Fig.34):

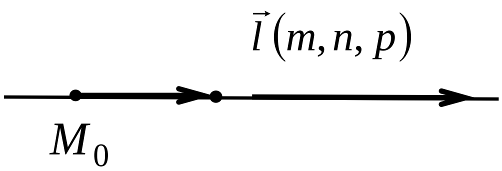


Figure 34

$$\overrightarrow{M_0M} \parallel \vec{l} \Leftrightarrow \boxed{\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}}$$

These equations are called *the canonical equations of the straight line*.

Note. One or two coordinates of the direction vector could be equal to zero. Canonical equations just show the proportionality of the coordinates of collinear vectors.

Example. Find the canonical equations of the straight line passing through the point $M_0(1,-3,0)$ and perpendicular to the vectors $\vec{a}(1,0,-3)$ and $\vec{b}(2,1,5)$.

The direction vector of this straight line is perpendicular to these vectors as well and therefore

$$\vec{l} = \lambda \vec{a} \times \vec{b} .$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3 \\ 2 & 1 & 5 \end{vmatrix} = 3\vec{i} - 11\vec{j} + \vec{k} = (3,-11,1).$$

Let $\vec{l} = 1 \cdot \vec{a} \times \vec{b} = (3,-11,1)$. Then the canonical equations of this straight line are

$$\frac{x-1}{3} = \frac{y-(-3)}{-11} = \frac{z-0}{1} \text{ or } \frac{x-1}{3} = \frac{y+3}{-11} = \frac{z}{1}.$$

2.2.9. Equation of Straight Line Passing through Two Points

Suppose we know the coordinates of two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ of the given straight line. Then

$$\vec{l} = (m, n, p) = \overrightarrow{M_1 M_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

and the canonical equations of this straight line have the following form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Example. Find the canonical equations of the straight line passing through the points $M_1(1,-3,0)$ and $M_2(2,-1,4)$. By the given formula we have

$$\frac{x-1}{2-1} = \frac{y-(-3)}{-1-(-3)} = \frac{z-0}{4-0} \text{ or } \frac{x-1}{1} = \frac{y+3}{2} = \frac{z}{4}.$$

2.2.10. Parametric Equations of Straight Line

From the canonical equations of straight line It follows that for any point of the straight line three values are equal to the same value (value of some parameter).

Let us denote this value as *t*. Then

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}=t\in R \quad \Leftrightarrow \quad \begin{cases} x=mt+x_0 \\ y=nt+y_0 \quad t\in R \\ z=pt+z_0 \end{cases}$$

The last equations are called *the parametric equations of the straight line*.

Note. Parameter *t* plays a role of continuous index by means of which all points of straight line are numbered.

Example. Find the canonical equations of the straight line passing through the point $M_0(1;2;1)$ and parallel to the vector $\vec{a}(1,0,-3)$. Vector $\vec{a}$ can be taken as the direction vector of this straight line and therefore we have

$$\begin{cases} x=1t+1=t+1 \\ y=0t+2=2 \qquad t\in R. \\ z=-3t+1 \end{cases}$$

2.2.11. Distance from Point to Straight Line

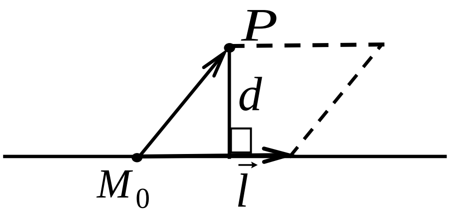


Figure 35

To find distance from the point $P(x_P,y_P,z_P)$ to the straight line It is enough to find the altitude of the parallelogram constructed on vectors $\vec{l}$ and $\overrightarrow{M_0P}$, where $\vec{l}=(m,n,p)$ is the direction vector of the straight line, $M_0(x_0,y_0,z_0)$ is any point of this straight line (Fig.35). Therefore

$$d=\frac{\left|\vec{l}\times\overrightarrow{M_0P}\right|}{\left|\vec{l}\right|}.$$

Example. Find the distance from the origin to the straight line $\frac{x-1}{1}=\frac{y+3}{2}=\frac{z}{4}$. Here

$$P(0,0,0),\ \vec{l}=(1,2,4),\ M_0(1,-3,0),\ \overrightarrow{M_0P}=(-1,3,0),$$

$$\left|\vec{l}\right| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1+4+16} = \sqrt{21},$$

$$\vec{l} \times \overrightarrow{M_0P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -1 & 3 & 0 \end{vmatrix} = -12\vec{i} - 4\vec{j} + 5\vec{k} = (-12,-4,5),$$

$$\left|\vec{l} \times \overrightarrow{M_0P}\right| = \sqrt{(-12)^2 + (-4)^2 + 5^2} = \sqrt{144+16+25} = \sqrt{185}.$$

Thus

$$d = \frac{\left|\vec{l} \times \overrightarrow{M_0P}\right|}{\left|\vec{l}\right|} = \sqrt{\frac{185}{21}}.$$

2.2.12. Positional Relationship of Straight Lines in Space

Definition. An angle between two straight lines is the angle between their direction vectors.

As It is shown on Fig.36

$$\cos\alpha = \frac{\left(\vec{l}_1, \vec{l}_2\right)}{\left|\vec{l}_1\right|\left|\vec{l}_2\right|}$$

then

$$\alpha = \arccos\frac{\left(\vec{l}_1, \vec{l}_2\right)}{\left|\vec{l}_1\right|\left|\vec{l}_2\right|}.$$

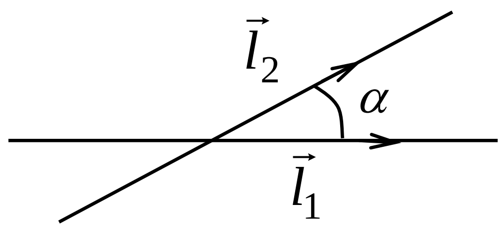


Fig.36

Moreover, for any two straight lines L_1, L_2 we have:

$$L_1 \parallel L_2 \Leftrightarrow \vec{l}_1 \parallel \vec{l}_2 \Leftrightarrow \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2};$$

$$L_1 \perp L_2 \Leftrightarrow \vec{l}_1 \perp \vec{l}_2 \Leftrightarrow m_1m_2 + n_1n_2 + p_1p_2 = 0.$$

Note. The angle between two straight lines does not give to us full information about positional relationship between these straight lines.

We have three different situations, namely:

a) crossing straight lines (Fig.37*a*);

b) parallel or coinciding straight lines (Fig.37*b*);

c) skew straight lines (Fig.37*c*).

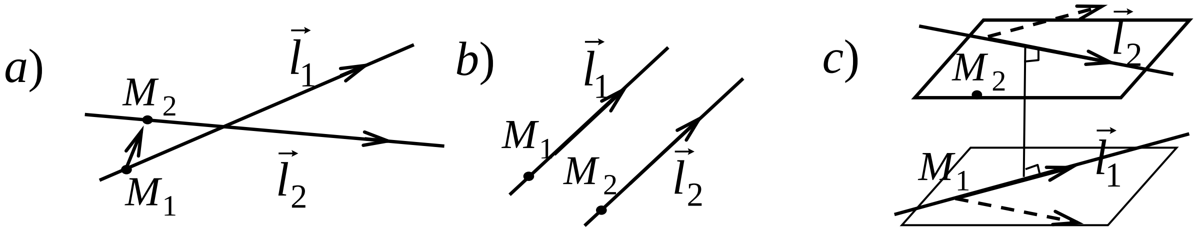


Figure 37

In cases *a*)-*b*) these straight lines lie in one plane, in case c) they lie in different planes.

It is simple to check that:

1) Straight lines lie in the same plane if and only if their direction vectors $\vec{l}_1, \vec{l}_2$ and vector $\overrightarrow{M_1M_2}$ connecting two different points of these straight lines are complanar, i.e.

$$\left(\vec{l}_1, \vec{l}_2, \overline{M_1M_2}\right) = 0 .$$

2) Two parallel lines coincide if and only if at least one point of one straight line belongs to other one.

It follows from discussed above that:

$$\left(\vec{l}_1, \vec{l}_2, \overline{M_1M_2}\right) \neq 0 \iff \text{straight lines are skew straight lines;}$$

$$\left(\vec{l}_1, \vec{l}_2, \overline{M_1M_2}\right) = 0 \iff \begin{cases} \text{straight lines are crossing, if } \vec{l}_1 \times \vec{l}_2 \neq 0 \\ \text{straight lines are parallel, if } \vec{l}_1 \times \vec{l}_2 = 0 \end{cases}$$

Example. Determine the positional relationship of two straight lines $\frac{x}{1} = \frac{y}{2} = \frac{z-1}{2}$ and $\frac{x-1}{-1} = \frac{y-3}{2} = \frac{z}{2}$. Since the direction vectors of these straight lines $\vec{l}_1(1,2,2)$, $\vec{l}_2(-1,2,2)$ are not collinear, these straight lines are not parallel.

Let us find the angle between them and determine either they are intersecting straight lines or skew ones.

$$\alpha = \arccos\frac{\left(\vec{l}_1, \vec{l}_2\right)}{\left|\vec{l}_1\right|\left|\vec{l}_2\right|} = \arccos\frac{-1+4+4}{\sqrt{1+4+4}\sqrt{1+4+4}} = \arccos\frac{7}{9}.$$

Here

$$M_1(0,0,1),\ M_2(1,3,0),$$

$$\overrightarrow{M_1M_2} = (1,3,-1),$$

$$\left(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}\right) = \begin{vmatrix} 1 & 2 & 2 \\ -1 & 2 & 2 \\ 1 & 3 & -1 \end{vmatrix} = -2+4-6-4-2-6 = -16 \neq 0.$$

Therefore, these straight lines are skew straight lines.

2.2.13. Distance between Two Straight Lines

Case 1. Suppose we have two crossing straight lines (Fig.37*a*). Then the distance between them is equal to zero, i.e.

$$d = 0.$$

Case 2. Suppose we have two parallel or coinciding straight lines (Fig.37*b*). Then the distance between them is equal to the distance from any point of one straight line to other straight line. Thus it can be calculated by formula:

$$d = \frac{\left|\overrightarrow{M_1M_2} \times \vec{l}_1\right|}{\left|\vec{l}_1\right|}.$$

Case 3. Suppose we have two skew straight lines (Fig.37*c*). Let us plot from the points M_1 and M_2 direction vectors $\vec{l}_1, \vec{l}_2$ of both straight lines as it shown on Fig. 37*c*. We have formed two parallel planes. Common perpendicular of straight lines is perpendicular to these planes. Therefore to find distance between two skew straight lines It is enough to find distance between two parallel planes or just the altitude of the parallelepiped constructed on vectors $\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}$. Thus

$$d = \frac{\left|\left(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}\right)\right|}{\left|\vec{l}_1 \times \vec{l}_2\right|}.$$

Note. If two straight lines intersect then the numerator of the fraction in the last formula is equal to zero, while the denominator is not equal to zero. Therefore, we can use the last formula to find distance between any two unparallel straight lines (either intersecting or skew ones).

Example. Find the distance between two straight lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z-1}{2} \text{ and } \frac{x-1}{-1} = \frac{y-3}{2} = \frac{z}{2}.$$

Since the direction vectors of these straight lines $\vec{l}_1(1,2,2)$, $\vec{l}_2(-1,2,2)$ are not collinear, these straight lines are not parallel. So we should use the last formula to find distance. Here

$$M_1(0,0,1),\ M_2(1,3,0),\ \overrightarrow{M_1M_2} = (1,3,-1),$$

$$\vec{l}_1 \times \vec{l}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ -1 & 2 & 2 \end{vmatrix} = 0\vec{i} - 4\vec{j} + 4\vec{k} = (0,-4,4),$$

$$\left|\vec{l}_1 \times \vec{l}_2\right| = \sqrt{0^2 + (-4)^2 + 4^2} = \sqrt{2 \cdot 16} = 4\sqrt{2},$$

$$\left(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}\right) = \left(\vec{l}_1 \times \vec{l}_2, \overrightarrow{M_1M_2}\right) = 0 - 12 - 4 = -16.$$

Therefore we obtain

$$d = \frac{\left|\left(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}\right)\right|}{\left|\vec{l}_1 \times \vec{l}_2\right|} = \frac{16}{4\sqrt{2}} = 2\sqrt{2}.$$

2.2.14. Equation of Common Perpendicular to Skew Straight Lines

Since a direction vector of common perpendicular is perpendicular to both straight lines, we have

$$\vec{l} = \vec{l}_1 \times \vec{l}_2.$$

Common perpendicular is a straight line from intersection of two planes, namely (Fig.38):

1) plane passing trough the first straight line and parallel to $\vec{l}$;

2) plane passing through the second straight line and parallel to $\vec{l}$.

It follows from above that general equation of the common perpendicular is

$$\begin{cases} A_1(x-x_1)+B_1(y-y_1)+C_1(z-z_1)=0 \\ A_2(x-x_2)+B_2(y-y_2)+C_2(z-z_2)=0 \end{cases}$$

where

$$\bar{n}_1=(A_1,B_1,C_1)=\vec{l}_1\times\vec{l}\,,$$

$$\bar{n}_2=(A_2,B_2,C_2)=\vec{l}_2\times\vec{l}\,,$$

$M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ are points of the first and the second straight lines relatively.

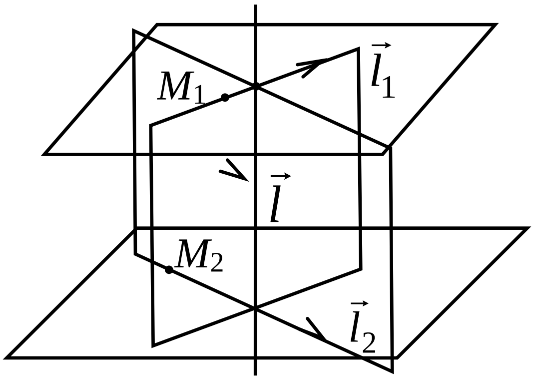


Figure 38

2.2.15. Positional Relationship of Plane and Straight Line

An angle α between straight line and a plane can be found in the following way (Fig. 39)

$$\alpha=\frac{\pi}{2}-\beta \text{ if } \beta=(\vec{l}\,\hat{,}\,\vec{n}) \text{ is acute (Fig.39}a\text{);}$$

$$\alpha=\beta-\frac{\pi}{2} \text{ if } \beta=(\vec{l}\,\hat{,}\,\vec{n}) \text{ is obtuse (Fig.39}b\text{).}$$

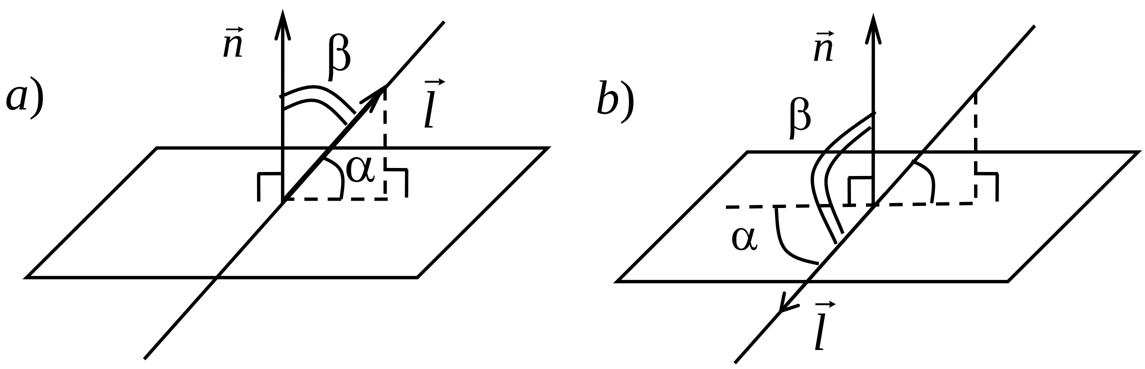


Figure 39

In other case,

$$\sin\alpha = |\cos\beta| = \left|\frac{(\vec{l},\vec{n})}{|\vec{l}||\vec{n}|}\right|,\ \alpha = \arcsin\left|\frac{(\vec{l},\vec{n})}{|\vec{l}||\vec{n}|}\right|.$$

From the last formula we have

1. Straight line is parallel to the plane $\Leftrightarrow \vec{l} \perp \vec{n} \Leftrightarrow (\vec{l},\vec{n}) = 0$;

2. Straight line is perpendicular to the plane $\Leftrightarrow$ $\vec{l} \parallel \vec{n} \Leftrightarrow \vec{l} \times \vec{n} = 0 \Leftrightarrow \frac{m}{A} = \frac{n}{B} = \frac{p}{C}$;

3. Straight line crosses a plane $\Leftrightarrow (\vec{l},\vec{n}) \neq 0$;

4. Straight line passing through the point M_0 belongs to the plane (Fig.40)$\Leftrightarrow$

1) $\vec{n} \perp \vec{l} \Leftrightarrow (\vec{n},\vec{l}) = 0 \Leftrightarrow Am + Bn + Cp = 0$

2) $M_0 : Ax_0 + By_0 + Cz_0 + D = 0$

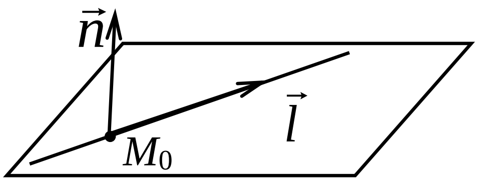


Figure 40

Example 1. Find an angle between the following straight line and plane: $\frac{x-1}{-2} = \frac{y}{3} = \frac{z-3}{4}$, $-x + 6y - 4z + 1 = 0$.

Here

$$\vec{l}(-2,3,4),\ \vec{n}(-1,6,-4).$$

Therefore,

$$\alpha = \arcsin\left|\frac{(\vec{l},\vec{n})}{|\vec{l}||\vec{n}|}\right| = \arcsin\left|\frac{2+18-16}{\sqrt{4+9+16}\sqrt{1+36+16}}\right| = \arcsin\frac{4}{\sqrt{29}\sqrt{53}} = \arcsin\frac{4}{\sqrt{1537}}.$$

Example 2. Prove that straight line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ lies in plane $-x + 6y - 4z + 1 = 0$. Here

$$M_0(1,2,3),\ \vec{l}(2,3,4),\ \vec{n}(-1,6,-4).$$

$(\vec{n},\vec{l}) = 2(-1) + 3\cdot 6 + 4(-4) = -2 + 18 - 16 = 0$.

$Ax_0 + By_0 + Cz_0 + D = -1\cdot 1 + 6\cdot 2 - 4\cdot 3 + 1 = -1 + 12 - 12 + 1 = 0$.

Thus, $\vec{n} \perp \vec{l}$ and M_0 belongs to this plane, i.e. this straight line lies in plane.

2.2.16. Point of Intersection of Straight Line and Plane

To find this point means to find solution of the following system:

$$\begin{cases} Ax+By+Cz+D=0 \\ \dfrac{x-x_0}{mn}=\dfrac{y-y_0}{n}=\dfrac{z-z_0}{p} \end{cases} \Leftrightarrow \begin{cases} Ax+By+Cz+D=0 \\ x=mt+x_0 \\ y=nt+y_0 \\ z=pt+z_0 \end{cases}$$

For this it is enough to substitute expressions for *x*, *y*, *z* into the equation of the plane and solve one equation for parameter *t*.

There are three possible situations:

1) The only solution $\Leftrightarrow$ the only point of intersection $\Leftrightarrow$ straight line crosses this plane;
2) No solutions $\Leftrightarrow$ no points of intersection $\Leftrightarrow$ straight line is parallel to this plane;
3) A lot of solutions $\Leftrightarrow$ a lot of points of intersection $\Leftrightarrow$ straight line belongs to this plane.

Example. Find the point of intersection of the straight line $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ and plane $x+2y+3z-4=0$.

First, we write down the parametric equations of this straight line: $x=2t+1$, $y=3t+2$, $z=4t+3$, $t\in R$. Then we solve the system:

$$\begin{cases} x+2y+3z-4=0 \\ x=2t+1 \\ y=3t+2 \\ z=4t+3 \end{cases} \Leftrightarrow \begin{cases} (2t+1)+2(3t+2)+3(4t+3)-4=20t+10=0 \\ x=2t+1 \\ y=3t+2 \\ z=4t+3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} t=-1/2 \\ x=0 \\ y=1/2 \\ z=1 \end{cases}$$

Therefore the point of intersection is $P(0,1/2,1)$.

2.3. Straight Line in Plane

2.3.1. Line in Plane

Consider the rectangular Cartesian coordinate system.

Definition. Locus of the points with coordinates satisfying an equation $\Phi(x, y) = 0$ is a line in plane.

Definition. Expression $\Phi(x, y) = 0$ is called an equation of the line in plane if coordinates of all points of this line, and only of those points, satisfy the equation.

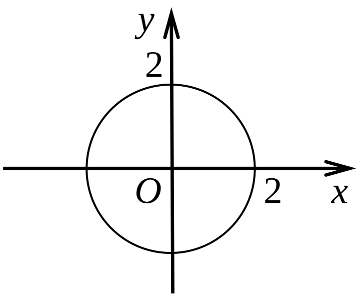


Figure 41

Example. Suppose

$$\Phi(x, y) = x^2 + y^2 - 4 = 0,$$

i.e.

$$x^2 + y^2 = 4 \text{ or } \sqrt{x^2 + y^2} = 2.$$

That is an equation of the circle with radius $R = 2$ (Fig.41).

Equation of the line depends on the system of coordinates. Let us consider, for example, *the polar system of coordinates*. Instead of to determine point in plane by Cartesian coordinates x, y we determine it by polar radius ρ and polar angle φ (Fig.42).

ρ is a distance from the point to the origin, i.e.

$$\rho = \left|\overrightarrow{OM}\right|;$$

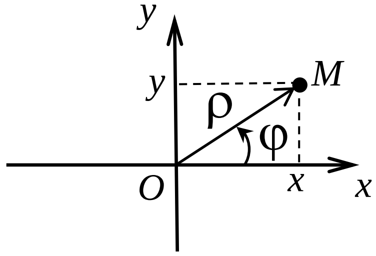


Figure 42

φ is an angle between the positive semi-axis Ox and the radius-vector of the point.

φ is positive in the anticlockwise direction and negative in other direction.

It is simple to get from Fig.42 that

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

Example. Find the equation of the circle with center (0;0) and radius $R=2$ in polar coordinates.

$$x^2+y^2=4 \Leftrightarrow \rho^2\cos^2\varphi+\rho^2\sin^2\varphi=4 \Leftrightarrow \rho^2(\cos^2\varphi+\sin^2\varphi)=4 \Leftrightarrow$$

$$\Leftrightarrow \rho^2=4 \Leftrightarrow \rho=2.$$

So, the equation of this circle in polar coordinates is

$$\rho=2.$$

Another way to determine a line in plane is by means of the parametric equations:

$$\begin{cases} x=x(t) \\ y=y(y) \end{cases} \quad t\in R$$

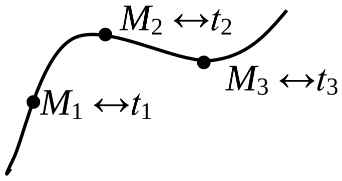


Figure 43

Here the coordinates of the points are functions of some parameter *t*. Parameter *t* is like a continuous number (index) of infinite number of points of this line (Fig.43).

Example. $\rho=2 \Rightarrow \begin{cases} x=2\cos\varphi \\ y=2\sin\varphi \end{cases}$ where φ is a parameter.

2.3.2. Straight Line in Plane

General Equation of the Straight Line

Consider the rectangular Cartesian coordinate system.

Theorem (about general equation of the straight line) Equation $Ax+By+C=0$, where $A^2+B^2\neq 0$, is an equation of the straight line in plane.

Proof. Let us consider an equation

$$Ax+By+C=0 \qquad (*)$$

and suppose that $x=x_0, y=y_0$ satisfy the equation (*). i.e.

$$Ax_0+By_0+C=0. \qquad (**)$$

The difference of equations (*) and (**) is

$$A(x-x_0)+B(y-y_0)=0. \qquad (***)$$

Equation (***) means that for any point $M(x,y)$ with coordinates satisfying the equation (*) the vector $\overrightarrow{M_0M} \perp \vec{n}$, where $M_0(x_0, y_0)$, $\vec{n} = (A, B)$ (Fig.44). Thus M belongs to the straight line passing through the point M_0 and perpendicular to the vector $\vec{n}$. Note, that from (***) we have

$$Ax - Ax_0 + By - By_0 = 0 \Leftrightarrow Ax + By + (-Ax_0 - By_0) = 0 \Leftrightarrow Ax + By + D = 0,$$

where $D = -Ax_0 - By_0$.

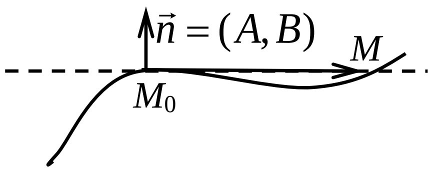


Figure 44

Let us consider now any point $M(x,y)$ that belongs to this straight line. Vector $\overrightarrow{M_0M}$ is the vector parallel to this straight line and therefore it is perpendicular to the vector $\vec{n} = (A, B)$, i.e.

$$0 = \left(\overrightarrow{M_0M}, \vec{n}\right) = A(x - x_0) + B(y - y_0) = Ax + By + (-Ax_0 - By_0) = Ax + By + D.$$

So, any point of this straight line satisfies the equation (*). ***Theorem is proven.***

Definition. Vector $\vec{n} = (A, B)$ is called a normal vector of the straight line $Ax + By + C = 0$.

Definition. Equation $Ax + By + C = 0$ is called *the general equation of the straight line*.

Definition. Equation $A(x - x_0) + B(y - y_0) = 0$ is called the equation of the straight line passing through the point $M_0(x_0, y_0)$ with normal vector $\vec{n}(A, B)$.

Example. Find the equation of the straight line parallel to the straight line $2x + y + 1 = 0$ and passing through the point $M_0(3, -2)$. Normal vector of the given straight line $\vec{n}(2,1)$ is perpendicular to the asked straight line as well, since these straight lines are parallel. Therefore we can take $\vec{n}(2,1)$ as normal vector of the straight line. So, we obtain

$$A(x - x_0) + B(y - y_0) = 2(x - 3) + 1(y - (-2)) = 2x - 6 + y + 2 = 2x + y - 4 = 0$$

or

$$2x + y - 4 = 0.$$

2.3.3. Equation of the Straight Line with Given Intercepts

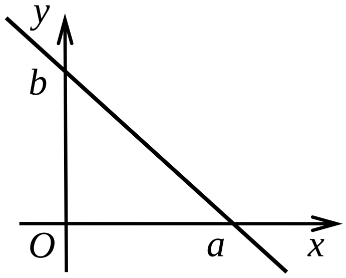


Figure 45

Suppose, $A \cdot B \cdot C \neq 0$. Let us divide the general equation of the straight line by C. Then

$$Ax + By + C = 0 \Leftrightarrow \frac{A}{C}x + \frac{B}{C}y + 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \Leftrightarrow \frac{x}{a} + \frac{y}{b} = 1,$$

where $a = -\frac{C}{A}$, $b = -\frac{C}{B}$ are the intercepts of the straight line on the exes (Fig.45).

Note. Intercepts a, b can be negative. It means that segments are cut from the negative semi-axes of *Ox*, *Oy*.

Example. Find the area of the triangle bounded by axes O*x*, O*y* and the straight line $\frac{x}{6} - \frac{y}{5} = 1$ (Fig.45). From the equation we have

$$a = 6,\ b = -5.$$

Therefore

$$S = \frac{1}{2}|a||b| = \frac{1}{2} \cdot 6 \cdot 5 = 15 \text{ square units.}$$

2.3.4. Canonical Equation of the Straight Line

As it was mentioned above, the straight line can be determined by its normal vector giving an orientation of the straight line and by its point giving a location of this straight line. But besides the normal vector the orientation of the straight line in plane can be given by the direction vector $\vec{l} = (m, n)$ (Fig.46). In this case

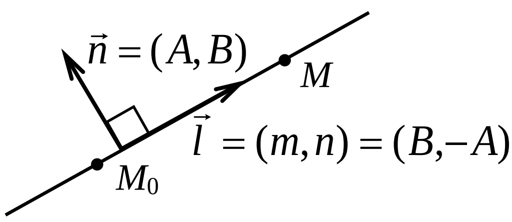


Figure 46

$$\vec{l} \parallel \overrightarrow{M_0M} \Leftrightarrow \frac{x-x_0}{m} = \frac{y-y_0}{n}.$$

The last equation is called *the canonical equation of the straight line in plane*.

Example. Find the equation of the straight line perpendicular to the straight line $2x+y+1=0$ and passing through the point $M_0(3,-2)$. Normal vector of the given straight line $\vec{n}(2,1)$ is parallel to the asked straight line, since these straight lines are perpendicular. Therefore we can take $\vec{n}(2,1)$ as the direction vector of the straight line. So, we obtain

$$\frac{x-x_0}{m} = \frac{y-y_0}{n} \Leftrightarrow \frac{x-3}{2} = \frac{y-(-2)}{1} \Leftrightarrow x-3 = 2(y+2) \Leftrightarrow$$

$$\Leftrightarrow x-3 = 2y+4 \Leftrightarrow x-2y-3-4=0$$

or

$$x-2y-7=0.$$

2.3.5. Canonical Equation of the Straight Line Passing Through Two Points

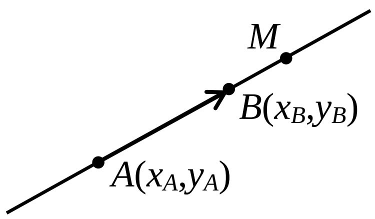


Figure 47

Suppose the points A and B belong to the straight line. Let us consider an arbitrary point of this line $M(x,y)$ (Fig.47). Then

$$\overrightarrow{AM} \parallel \overrightarrow{AB} \Leftrightarrow \frac{x-x_A}{x_B-x_A} = \frac{y-y_A}{y_B-y_A}.$$

The last equation is the equation of the straight line passing through two points A and B.

Example. Find the equation of the straight line passing through the origin and the point $B(1;-1)$. In that case the second point is $A(0;0)$. So we obtain

$$\frac{x-0}{1-0} = \frac{y-0}{-1-0} \Leftrightarrow x = -y.$$

Therefore the equation of the asked straight line is

$$x+y=0.$$

2.3.6. Parametric Equations of the Straight Line

Let us consider the canonical equation of the straight line and equate both sides of this equation to some value *t*:

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=t\,.$$

t is parameter of proportionality for point $M(x,y)$. From the last equation we get *the parametric equations of the straight line*:

$$\begin{cases} x = mt + x_0 \\ y = nt + y_0 \end{cases} \quad t \in R\,.$$

2.3.7. Distance between Point and Straight Line

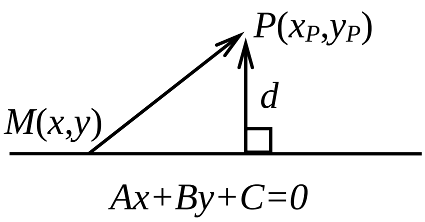


Figure 48

Distance from the point $P(x_P,y_P)$ to the straight line can be found as a module of the projection of the vector $\overrightarrow{MP}$ on normal direction (Fig.48):

$$d=\left|pr_{\vec{n}}\overrightarrow{MP}\right|=\left|\frac{\left(\vec{n},\overrightarrow{MP}\right)}{|\vec{n}|}\right|=\frac{|A(x_P-x)+B(y_P-y)|}{\sqrt{A^2+B^2}}=$$

$$=\frac{|Ax_P+By_P-(Ax+By)|}{\sqrt{A^2+B^2}}=\frac{|Ax_P+By_P+C|}{\sqrt{A^2+B^2}}\,.$$

So, the distance from the point $P(x_P,y_P)$ to the straight line $Ax+By+C=0$ is

$$d=\frac{|Ax_P+By_P+C|}{\sqrt{A^2+B^2}}\,.$$

Example 1. Find the distance from the origin to the straight line:

$$d_0=\frac{|Ax_p+By_p+C|}{\sqrt{A^2+B^2}}=\begin{bmatrix} x_p=0 \\ y_p=0 \end{bmatrix}=\frac{|A\cdot 0+B\cdot 0+C|}{\sqrt{A^2+B^2}}=\frac{|C|}{\sqrt{A^2+B^2}}\,.$$

Example 2. Find the altitude of the triangle with vertices $A(1;-2),\ B(2;0),\ C(-1;3)$ dropped on the side *AC* (Fig.49).

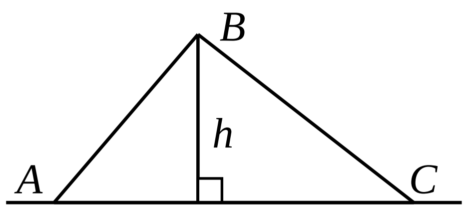


Figure 49

The value of this altitude can be found as a distance from the point *B* to the straight line, passing through the points *A* and *C*.

The equation of the straight line *AC* is

$$\frac{x-1}{-1-1}=\frac{y-(-2)}{3-(-2)} \Leftrightarrow 5(x-1)=-2(y+2) \Leftrightarrow$$

$$\Leftrightarrow 5x-5=-2y-4 \Leftrightarrow 5x+2y-5+4=0 \Leftrightarrow 5x+2y-1=0.$$

Therefore

$$h=d=\frac{|5\cdot 2+2\cdot 0-1|}{\sqrt{5^2+2^2}}=\frac{9}{\sqrt{29}}.$$

2.3.8. Normal Equation of the Straight Line

Let us consider the general equation of the straight line and divide it by the module of its normal vector:

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y+\frac{C}{\sqrt{A^2+B^2}}=0 \Leftrightarrow \cos\alpha\, x+\cos\beta\, y+p=0,$$

where $\cos\alpha=\frac{A}{\sqrt{A^2+B^2}}, \cos\beta=\frac{B}{\sqrt{A^2+B^2}}$ are direction cosines of normal vector, i.e. coordinates of the ort $\vec{n}^0$,

$p=\frac{C}{\sqrt{A^2+B^2}}$ is a distance from the origin to this straight line taken with sign equal to the sign of the coefficient *C*.

The obtained equation

$$\cos\alpha\, x+\cos\beta\, y+p=0$$

is called the normal equation of this straight line.

Note. By means of the direction cosines we can rewrite the formula for point distance to straight line in the following way:

$$d=\frac{|Ax_p+By_p+C|}{\sqrt{A^2+B^2}}=$$

$$=\left|\frac{A}{\sqrt{A^2+B^2}}x_p+\frac{B}{\sqrt{A^2+B^2}}y_p+\frac{C}{\sqrt{A^2+B^2}}\right|=\left|\cos\alpha\ x_P+\cos\beta\ y_P+p\right|,$$

i.e.

$$d=\left|\cos\alpha\ x_P+\cos\beta\ y_P+p\right|.$$

Example. Find the equations of straight lines perpendicular to the vector $\vec{n}(3,4)$ and located on the distance 3 from the origin.

Since

$$|\vec{n}|=\sqrt{3^2+4^2}=\sqrt{25}=5,\ \vec{n}^0\left(\frac{3}{5},\frac{4}{5}\right)=(\cos\alpha,\cos\beta),$$

the normal equations of these straight lines look like

$$\cos\alpha\, x+\cos\beta\, y+p=\frac{3}{5}x+\frac{4}{5}y+p=0.$$

From condition of the example It follows that

$$p=\pm 3$$

and therefore the asked equations are

$$\frac{3}{5}x+\frac{4}{5}y\pm 3=0 \Leftrightarrow 3x+4y\pm 15=0.$$

2.3.9. Equation of the Straight Line with Given Slope

Another way to determine the straight line is by the given slope and y-intercept (the segment cut from the axis Oy).

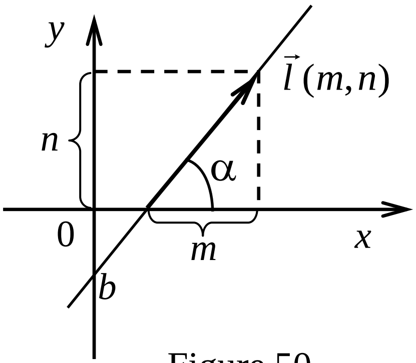


Figure 50

Definition. Slope is a tangent of the angle between the straight line and the positive semi-axis Ox.

Notation: k.

So (Fig.50),

$$k=\tan\alpha.$$

Let us obtain the equation of this straight line:

$$\frac{x-0}{m}=\frac{y-b}{n} \Leftrightarrow y-b=\frac{n}{m}x \Leftrightarrow$$

$$y=\frac{n}{m}x+b=\tan\alpha\cdot x+b \Leftrightarrow y=kx+b.$$

We obtained the equation of *the straight line with the given slope k and the intercept b:*

$$y=kx+b\,. \tag{*}$$

Suppose (x_0, y_0) is a point of this straight line. Then

$$y_0=kx_0+b\,. \tag{**}$$

The difference of the equations (*) and (**) gives *the equation of the straight line with slope k passing through the point* (x_0, y_0):

$$y-y_0=k(x-x_0)\,.$$

If you know any two points of the straight line $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ then:

$$\vec{l}=\overrightarrow{M_1M_2}=(x_2-x_1, y_2-y_1)=(m,n),$$

$$k=\frac{n}{m}=\frac{y_2-y_1}{x_2-x_1}\,.$$

Example. Find the equations of the triangle sides if the vertices of this triangle are $A(1,-2),\ B(2,0),\ C(-1,3)$.

Let us find the corresponding slops and equations:

$$k_{AB}=\frac{y_B-y_A}{x_B-x_A}=\frac{0-(-2)}{2-1}=2\,,$$

Side AB: $y-y_A=k_{AB}(x-x_A) \Leftrightarrow y-(-2)=2(x-1) \Leftrightarrow 2x-y-4=0$;

$$k_{AC}=\frac{y_C-y_A}{x_C-x_A}=\frac{3-(-2)}{-1-1}=-\frac{5}{2},$$

Side AC: $y-y_A=k_{AC}(x-x_A) \Leftrightarrow y-(-2)=-\frac{5}{2}(x-1) \Leftrightarrow 5x+2y-1=0$;

$$k_{CB}=\frac{y_B-y_C}{x_B-x_C}=\frac{0-3}{2-(-1)}=-1\,,$$

Side AB: $y-y_B=k_{CB}(x-x_B) \Leftrightarrow y-0=-1(x-2) \Leftrightarrow x+y-2=0$.

2.3.10. Angle between Two Straight Lines

An angle between two straight lines can be found in three ways: as angle between their normal vectors $\vec{n}_1, \vec{n}_2$, as angle between their direction vectors $\vec{l}_1, \vec{l}_2$ and by means of their slopes k_1, k_2.

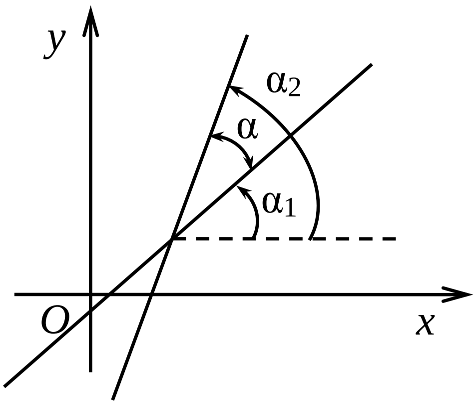


Figure 51

In the first two cases we calculate the angle by formulas known from vector algebra:

$$\cos\alpha = \frac{(\vec{l}_1, \vec{l}_2)}{|\vec{l}_1||\vec{l}_2|} \text{ or } \cos\alpha = \frac{(\vec{n}_1, \vec{n}_2)}{|\vec{n}_1||\vec{n}_2|}.$$

To get acute (or right) angle between two straight lines use formulas:

$$\cos\alpha = \left|\frac{(\vec{l}_1, \vec{l}_2)}{|\vec{l}_1||\vec{l}_2|}\right| \text{ or } \cos\alpha = \left|\frac{(\vec{n}_1, \vec{n}_2)}{|\vec{n}_1||\vec{n}_2|}\right|.$$

Let us consider the last variant (Fig.51).

$$k_1 = \tan\alpha_1, k_2 = \tan\alpha_2, \alpha = \alpha_2 - \alpha_1.$$

$$\tan\alpha = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{k_2 - k_1}{1 + k_1 k_2}.$$

To get acute (or right) angle between two straight lines use formula:

$$\tan\alpha = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right|.$$

Note 1. There are always two positive angles between two straight lines, namely α and $\pi - \alpha$ (Fig.52).

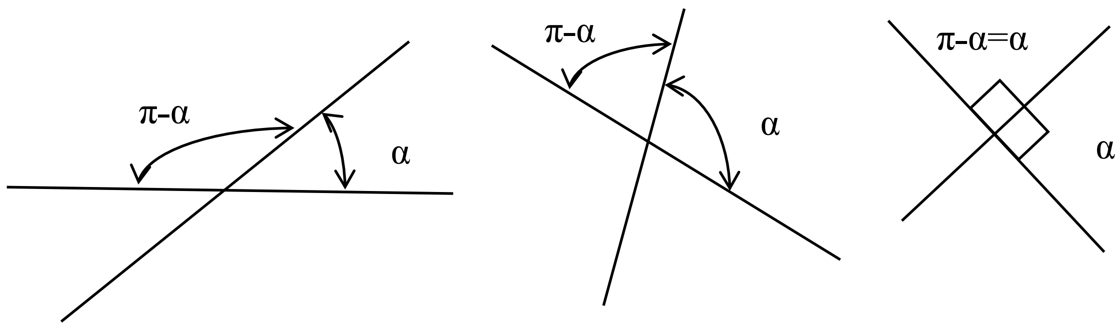


Figure 52

Note 2. From the obtained formulas we get some conditions for special positional relationships of straight lines:

1. *Two straight lines are perpendicular if and only if*

$$\vec{n}_1 \perp \vec{n}_2,\ \text{i.e. } (\vec{n}_1, \vec{n}_2) = 0;$$

or $$\vec{l}_1 \perp \vec{l}_2,\ \text{i.e. } (\vec{l}_1, \vec{l}_2) = 0;$$

or $$\cot\alpha = 0,\ \text{i.e. } 1 + k_1 k_2 = 0 \text{ or } k_2 = \frac{-1}{k_1}.$$

2. *Two straight lines are parallel if and only if*

$$\vec{n}_1 \parallel \vec{n}_2,\ \text{i.e. } \frac{A_1}{A_2} = \frac{B_1}{B_2};$$

or $$\vec{l}_1 \parallel \vec{l}_2,\ \text{i.e. } \frac{m_1}{m_2} = \frac{n_1}{n_2};$$

or $$\tan\alpha = 0,\ \text{i.e. } k_2 - k_1 = 0 \text{ or } k_1 = k_2.$$

Example. Find the equations of the straight lines parallel and perpendicular to the straight line $2x+y-3=0$ if they pass through the point $A(3;0)$ (Fig.53).

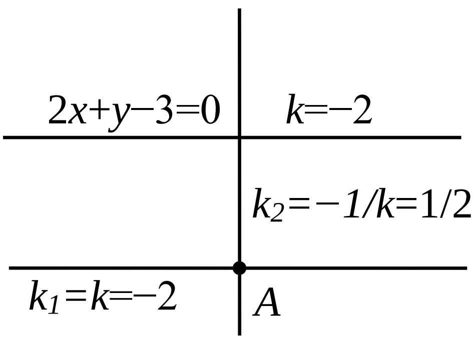


Figure 53

The slope of the given straight line is equal to the coefficient of x when y is expressed from the equation:

$$y = -2x + 3 \Rightarrow k = -2.$$

Since the first straight line is parallel to the initial straight line then its slope

$$k_1=k=-2.$$

Therefore the equation of the first straight line is

$$y - y_A = k_1(x - x_A) \Leftrightarrow y - 0 = -2(x-3) \Leftrightarrow 2x + y - 6 = 0.$$

Since the second straight line is perpendicular to the initial straight line then its slope

$$k_2=-1/k=1/2.$$

Therefore the equation of the first straight line is

$$y - y_A = k_2(x - x_A) \Leftrightarrow y - 0 = 1/2 \cdot (x-3) \Leftrightarrow x - 2y - 3 = 0.$$

2.4. Curves of the Second Order in Plane

Definition. Locus of the points with coordinates satisfying the general equation of the second degree is the curve of the second order.
At the same time the equation

$$\Phi(x,y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where $A^2 + B^2 + C^2 \neq 0$, is called *the general equation of the second order curve*.

The second order curves are, for example, circle, ellipse, hyperbola, parabola, pair of straight lines, etc.

2.4.1. Circle

Definition. Circle is a locus of the point which moves so that its distance from a fixed point, called the center, is equal to a given distance (Fig.45). The given distance is called the radius of the circle

$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2} = R \text{ or}$$

$$(x-x_0)^2 + (y-y_0)^2 = R^2.$$

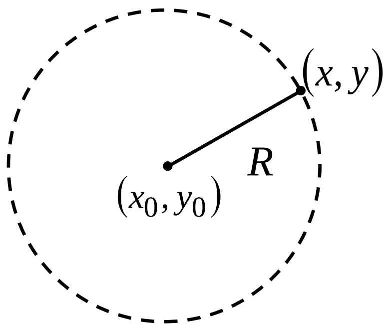


Figure 54

This equation is a canonical equation of the circle.

Distinguishing features of circle equation:

1. The coefficients of x^2 and y^2 are equal to each other.
2. The coefficients of xy is equal to zero.

Suppose we have equation

$$\Phi(x,y) = x^2 + y^2 + 2Dx + 2Ey + F = 0.$$

When will it be a circle equation?

First we should rearrange the terms and complete the square in both variables *x* and *y*.

Completing the square is the procedure that consists basically of adding and subtracting certain quantities to the second-degree equation to form the sum of two perfect squares. When both the first- and the second-degree members of the same variable are known, the square of one-half the coefficient of the first-degree term should be added and subtracted. This will allow the quadratic equation to be factored into the sum of two perfect squares.

Therefore the given above equation may be rewritten in the following way:

$$x^2+2Dx+D^2-D^2+y^2+2Ey+E^2-E^2+F=0$$

$$(x+D)^2+(y+E)^2=D^2+E^2-F$$

There are three different cases:

1) $D^2+E^2-F>0$.

Then this equation is the equation of the circle with the origin in the point $(-D,-E)$ and of the radius equal to $\sqrt{D^2+E^2-F}$.

2) $D^2+E^2-F=0 \Leftrightarrow$ It is a point $(-D,-E)$

3) $D^2+E^2-F<0 \Rightarrow$ Radius of the circle is imaginary. In this case the given equation does not represent any real geometrical locus.

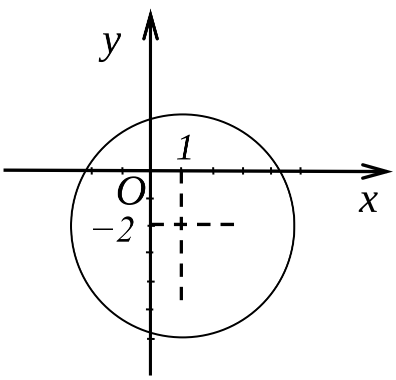


Figure 55

Example. Suppose we have the equation $x^2+y^2-2x+4y-11=0$. If that is equation of the circle then find the origin and the radius of this circle.

After completing the squares we obtain

$$(x-1)^2+(y+2)^2=16.$$

Thus, the origin is $(x_0,y_0)=(1,-2)$, the radius is $R=4$ (Fig.55).

2.4.2. Conic sections

Definition. The locus of a point *P*, which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line is called a *conic section*.

The fixed point is called the *Focus* (pl. focuses or foci) and is usually denoted by *F*.

The constant ratio is called the *Eccentricity* and is denoted by ε.

The fixed straight line is called the *Directrix*.

That is why this property of points of conic sections to save ratio of distances is called *the focal-directorial property* of the conic sections.

Definition 1. When $\varepsilon = 1$ the conic section is called *parabola*.

Definition 2. When $0 \le \varepsilon < 1$ the conic section is called *ellipse*.

Definition 3. When $\varepsilon > 1$ the conic section is called *hyperbola*.

Note 1. At $\varepsilon = 0$ the ellipse becomes the circle.

Note 2. The name Conic Section is derived from the fact that these curves were first obtained as plane sections of a right circular cone (Fig.56). A circle is formed when a cone is cut perpendicular to its axis. An ellipse is produced when

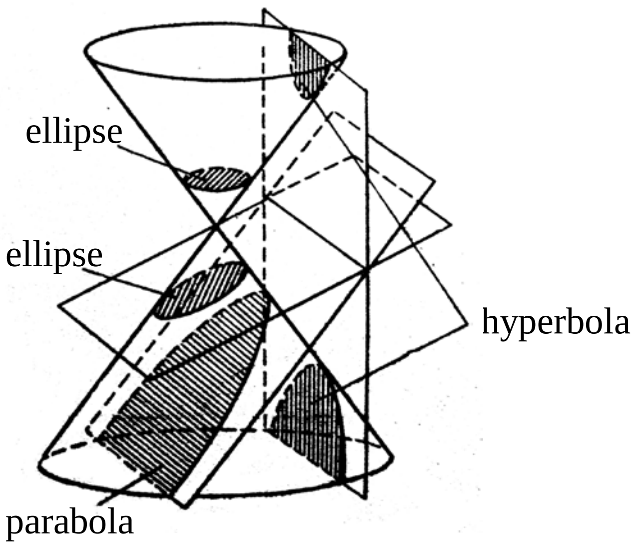


Figure 56

the cone is cut obliquely to the axis and the surface. A hyperbola results when the cone is intersected by a plane parallel to the axis, and a parabola results when the intersecting plane is parallel to an element of the surface.

Note 3. It will be shown later that conic sections are curves of the second order.

2.4.3. Canonical Equation of Parabola

Definition. Parabola is a locus of point with its distance from some fixed point *F* equal to its distance from some straight line.

It is clear that new definition is equivalent to the old one.

Point *F* is called a focus. Straight line is called a directrix.

To derive equation of the parabola we consider point $F(p/2,0)$ as focus and straight line $x=-p/2$ (Fig.57).

$$\varepsilon=\frac{r_1}{r_2}=1\Leftrightarrow r_1^2=r_2^2$$

$$r_1^2=\left(x-\frac{p}{2}\right)^2+(y-0)^2$$

$$r_2^2=\left(x+\frac{p}{2}\right)^2$$

$$x^2-px+\frac{p^2}{2^2}+y^2=x^2+px+\frac{p^2}{4}$$

$$\boxed{y^2=2px}$$

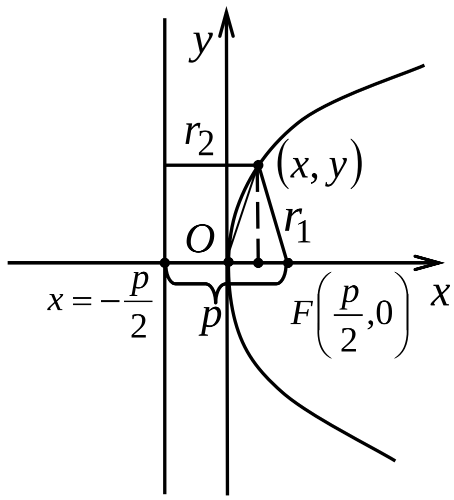


Figure 57

That is *a canonical equation of parabola*.

Properties of parabola graph:

1) Parabola is situated in a half plane with positive abscissa. Indeed,

$$y^2\ge 0\Leftrightarrow x\ge 0.$$

2) $|y|$ is increasing if *x* is increasing.

3) *Ox* is axis of symmetry. Indeed, for one value of *x* we have two values for *y* which differ only in sign.

If we change x by y and y by x in the canonical equation of parabola, then we get

$$\boxed{x^2 = 2py}$$

That is the canonical equation of parabola with axis of symmetry Oy (Fig.58).

Point O is called vertex of parabola.

Value p is distance between vertex and directrix.

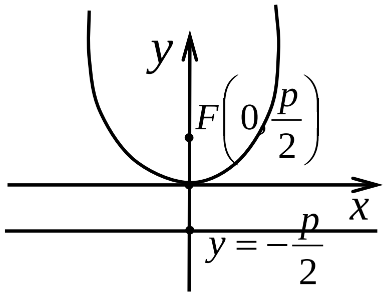


Figure 58

Note. If point (x_0, y_0) is a vertex of parabola and directrix is parallel to one of axes then canonical equation of parabola has form

$$(y-y_0)^2 = 2p(x-x_0) \text{ or } (x-x_0)^2 = 2p(y-y_0).$$

And new parabola is obtained by shift of

$$y^2 = 2px \quad \left(x^2 = 2py\right)$$

on x_0 along the axis Ox and on y_0 along the axis Oy.

Example. Reduce the equation of the parabola $x^2 + 2x + 4y - 7 = 0$ to the canonical form and plot the graph of this parabola.

After completing the squares we obtain

$$(x+1)^2 + 4y - 8 = 0 \text{ or } (x+1)^2 + 4(y-2) = 0 \text{ or}$$

$$(x+1)^2 = -4(y-2).$$

Thus, the vertex is $(x_0, y_0) = (-1,2)$ and $p = -2$. Graph of this parabola is presented on Fig.59.

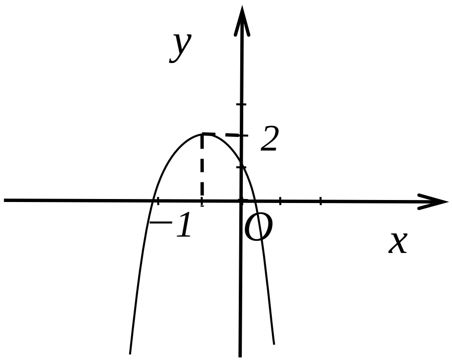


Figure 59

2.4.4. Canonical Equation of Ellipse

Definition. Ellipse is a locus of points with constant sum of distances from two fixed points called foci.

Let us find canonical equation of Ellipse.

Suppose, foci are in points $F_1(-c,0), F_2(c,0)$ and the sum of distances is equal to $2a$, where $a>c$ (Fig.60). Then

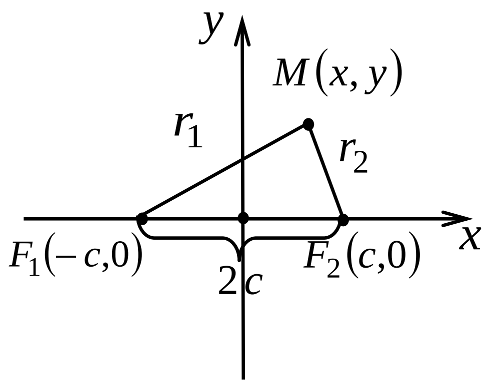


Figure 60

$$r_1 + r_2 = 2a\,,$$

$$r_1^2 = (x+c)^2 + y^2,$$

$$r_2^2 = (x-c)^2 + y^2,$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}\,,$$

$$\left(\sqrt{(x+c)^2 + y^2}\right)^2 = \left(2a - \sqrt{(x-c)^2 + y^2}\right)^2,$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2,$$

$$x^2 + 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2xc + c^2 + y^2,$$

$$2xc = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2xc\,,$$

$$4xc - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}\,,$$

$$\left(xc - a^2\right)^2 = \left(-a\sqrt{(x-c)^2 + y^2}\right)^2,$$

$$x^2c^2 - 2a^2xc - a^4 = a^2(x-c)^2 + a^2y^2,$$

$$x^2c^2 - 2a^2xc + a^4 = a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2,$$

$$a^4 - a^2c^2 = a^2x^2 - x^2c^2 + a^2y^2,$$

$$a^2\left(a^2 - c^2\right) = \left(a^2 - c^2\right)x^2 + a^2y^2,$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2}\,.$$

Let us denote by $b^2 = a^2 - c^2$, since $a>c$. Then the last equation has the following form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

called *the canonical equation of the ellipse*.

Note, that $a>b$. At that:
the value a is called *a major semi-axis*;
the value b is called *a minor semi-axis*.

Let us check that we do not have extraneous roots obtained at calculating the second power of expressions.

From canonical equation: $r_1 = \sqrt{(x+c)^2 + y^2} = \sqrt{(x+c)^2 + b^2\left(1 - \dfrac{x^2}{a^2}\right)} =$

$$= \sqrt{x^2\left(1 - \frac{b^2}{a^2}\right) + 2cx + a^2} = \sqrt{x^2\frac{c^2}{a^2} + 2cx + a^2} = \sqrt{\left(a + \frac{c}{a}x\right)^2} = \left|a + \frac{c}{a}x\right|.$$

But $\dfrac{c}{a} < 1$ and $\left(\dfrac{x^2}{a^2} \le 1 \Rightarrow |x| \le a\right)$.

Therefore, $r_1 = a + \dfrac{c}{a}x$.

In a similar way we get $r_2 = a - \dfrac{c}{a}x$.

Thus, $r_1 + r_2 = 2a$.

Note. If $a = b \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1 \Leftrightarrow x^2 + y^2 = a^2$, i.e. the ellipse with equal semi-axes is a circle.

$$c^2 = a^2 - b^2 = 0 \Leftrightarrow \text{For circle } F_1 = F_2 = (0,0)$$

Properties of ellipse graph:

1) Since variables in ellipse equation are squared then if $M(x, y)$ belongs to ellipse then points $(-x, y)$, $(x, -y)$, $(-x, -y)$ belong to ellipse, too. It means that ellipse has two axes of symmetry, namely *Ox* and *Oy*.

Center of symmetry is *a center* of the ellipse.

Points of intersections with axes O*x* and O*y* are *the vertices* of the ellipse.

$$(-a,0),(a,0),(0,b),(0,-b).$$

2) From canonical equation: $\frac{x^2}{a^2} \le 1$ and $\frac{y^2}{b^2} \le 1 \Rightarrow |x| \le a, \quad |y| \le b$.

It means that a graph of ellipse is situated inside the rectangle.

3) Let $\bar{x} = x$, $\bar{y} = \frac{a}{b} y$ be new variables.

If $\bar{x}^2 + \bar{y}^2 = a^2$, then $x^2 + \frac{a^2}{b^2} y^2 = a^2$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

It means that graph of ellipse can be obtained by pressing the circle in the direction of axis *Oy*.

Result of the analysis made above is presented on Fig.61.

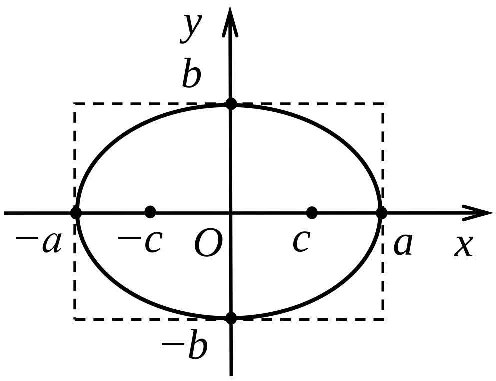


Figure 61

Eccentricity of ellipse ε can be found as the ratio of distance between foci and value of major axis, i.e.

$$\varepsilon = \frac{2c}{2a} = \frac{c}{a}.$$

From here it follows that

$$0 < \varepsilon < 1;$$
$$r_1 = a + \varepsilon x;$$
$$r_2 = a - \varepsilon x.$$

Two straight lines perpendicular to the major axis and situated symmetrically with distance a/ε from center are directrices of ellipse.

Let us show that *the focal-directorial property* is valid for such a definition of ellipse, i.e.

$$\frac{r}{d} = \varepsilon .$$

Since $\varepsilon < 1$, then $\frac{a}{\varepsilon} > a$. It means that directrices $x = \pm\frac{a}{\varepsilon}$ are situated outside the rectangle of ellipse. So (Fig.62),

$$\frac{r_1}{d_1} = \frac{a+\varepsilon x}{\frac{a}{\varepsilon}+x} = \frac{a+\varepsilon x}{\frac{a+\varepsilon x}{\varepsilon}} = \varepsilon;$$

$$\frac{r_2}{d_2} = \frac{a-\varepsilon x}{\frac{a}{\varepsilon}-x} = \frac{a-\varepsilon x}{\frac{a-\varepsilon x}{\varepsilon}} = \varepsilon.$$

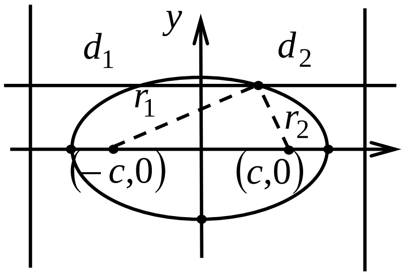


Figure 62

Thus, the property is valid.

Suppose that foci are situated on the axes *Oy* and the sum of distances from an ellipse point to foci is equal to 2*b*. Then in the similar way we can get:

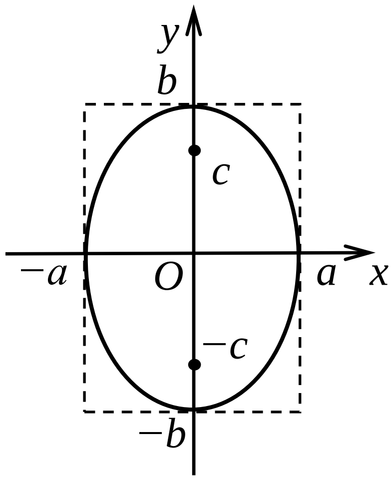


Figure 63

$$r_1 + r_2 = 2b \Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b>a$ (Fig.63), $c^2 = b^2 - a^2$, $\varepsilon = \frac{c}{b}$,

$r_1 = b + \varepsilon y$,

$r_2 = b - \varepsilon y$;

$y = \pm\frac{b}{\varepsilon}$ are directrices.

Here

a is called a minor semi-axis,

b is called a major semi-axis.

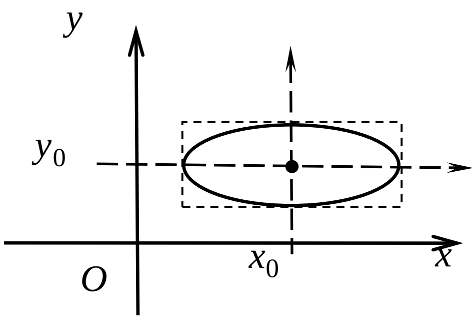


Figure 64

Note 1. If center of the ellipse is in the point (x_0, y_0) but ellipse axes are parallel to coordinate axes, then the canonical equation of this ellipse has the following form (Fig.64):

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1.$$

In this case,

$F_1(-c+x_0,\ 0+y_0)$, $F_2(c+x_0,\ 0+y_0)$ are foci,

$x = x_0 \pm \frac{a}{\varepsilon}$ are directrices.

Example. Plot the graphs of the following ellipses:

1) $\dfrac{x^2}{20}+\dfrac{y^2}{5}=1$; 2) $4x^2+y^2=36$; 3) $x^2-4x+4y^2+8y-28=0$.

The first ellipse is an ellipse with semi-axes $a=\sqrt{20}=2\sqrt{5}$, $b=\sqrt{5}$. Its graph is presented on Fig.65*a*.

To plot the second ellipse we should first reduce its equation to the canonical form dividing the equation by 36:

$$\frac{4x^2}{36}+\frac{y^2}{36}=\frac{36}{36}\Leftrightarrow\frac{x^2}{9}+\frac{y^2}{36}=1.$$

That is an ellipse with semi-axes $a=\sqrt{9}=3$, $b=\sqrt{36}=6$. Its graph is presented on Fig.65*b*.

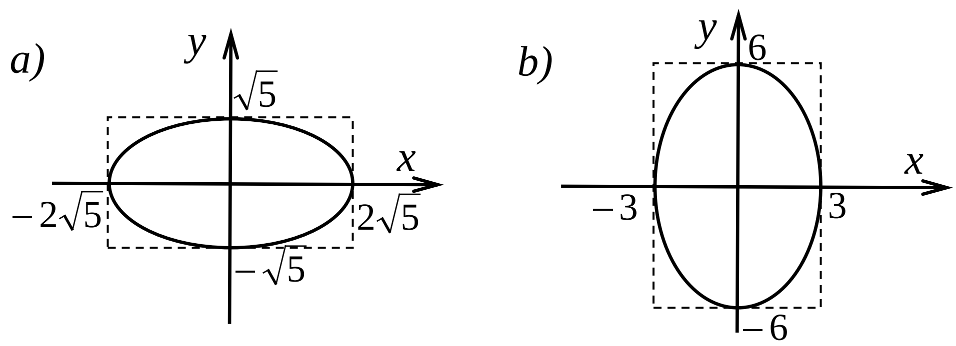


Figure 65

To plot the last ellipse we should complete the squares in x and y in the equation:

$$x^2-4x+4y^2+8y-28=x^2-4x+4\left(y^2+2y\right)-28=$$

$$=x^2-4x+4-4+4\left(y^2+2y+1-1\right)-28=$$

$$=(x-2)^2-4+4(y+1)^2-4-28=$$

$$=(x-2)^2+4(y+1)^2-36=0.$$

After transposing free term 36 to the right-hand side and dividing the equation by it we get:

$$\frac{(x-2)^2}{36}+\frac{(y+1)^2}{9}=1.$$

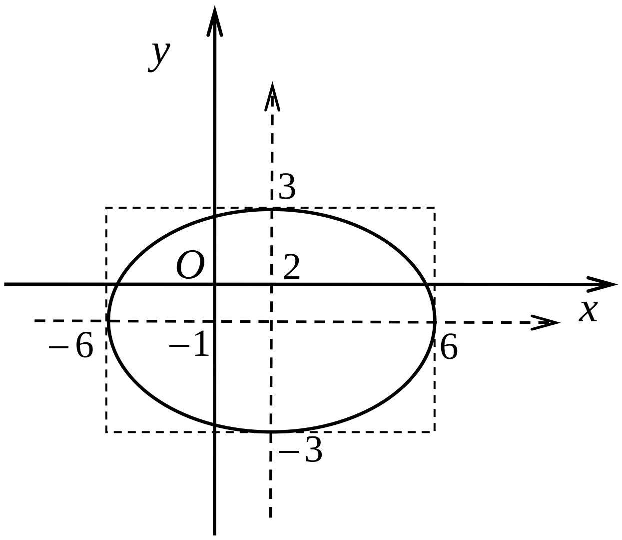


Figure 66

That is an ellipse with semi-axes $a=\sqrt{36}=6$, $b=\sqrt{9}=3$ and the vertex (2,−1). Its graph is presented on Fig.66.

Note 2. Canonical equation of the ellipse could be found directly from the initial definition, namely as a curve with $0\le\varepsilon<1$. Let us complete this derivation.

Suppose,

$PO=P'O=a$, $FO=c$, $MO=d_0$

where F is the focus, O is the center, P and P' are points of the ellipse situated on the straight line passing through focus and center, M is a point of directrix situated on the same straight line (Fig. 67). Then from the definition of eccentricity we have

$$\frac{a-c}{d_0-a}=\varepsilon \text{ or } a-c=\varepsilon d_0-\varepsilon a,$$

$$\frac{a+c}{d_0+a}=\varepsilon \text{ or } a+c=\varepsilon d_0+\varepsilon a.$$

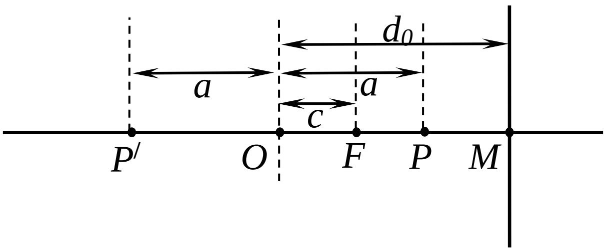


Figure 67

Subtraction and addition of these two equations give

$$2c=2a\varepsilon \text{ or } c=a\varepsilon,$$

$$2a=2d_0\varepsilon \text{ or } d_0=\frac{a}{\varepsilon}.$$

Note, that these formulas are true for any value of eccentricity.

Let us place the center of the ellipse at the origin so that the focus lies on the positive semi-axis Ox (Fig.68). Then for any arbitrary point (x,y) of ellipse we have

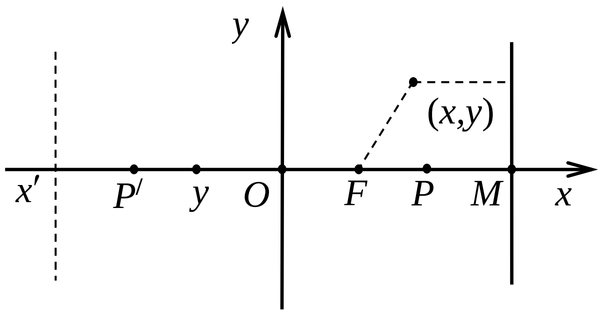


Figure 68

$$\varepsilon = \frac{\sqrt{(x-a\varepsilon)^2 + y^2}}{\frac{a}{\varepsilon} - x} \text{ or } \sqrt{(x-a\varepsilon)^2 + y^2} = a - x\varepsilon .$$

Squaring and expanding both sides give

$$x^2 - 2xa\varepsilon + a^2\varepsilon^2 + y^2 = a^2 - 2ax\varepsilon + x^2\varepsilon^2 .$$

Canceling like terms and transposing terms in x to the left-hand side of the equation give

$$x^2 - x^2\varepsilon^2 + y^2 = a^2 - a^2\varepsilon^2 .$$

Removing a common factor gives

$$x^2(1-\varepsilon^2) + y^2 = a^2(1-\varepsilon^2) .$$

Dividing both sides of this equation by the right-hand member gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-\varepsilon^2)} = 1 .$$

From equation we obtain y-intercept of ellipse

$$b^2 = a^2(1-\varepsilon^2) \text{ or } b = a\sqrt{1-\varepsilon^2} .$$

so that the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a \le b$.

This is the equation of an ellipse in the canonical form.

Note 3. In Note 2 we, actually, have found only the right branch of ellipse. The left branch could be found in the similar way but with the focus $F'(-a\varepsilon,0)$ and the directrix $x = -d_0 = -\frac{a}{\varepsilon}$. Equation of the left branch gives the same canonical equation.

2.4.5. Canonical Equation of Hyperbola

Definition. Hyperbola is locus of points with constant absolute value of difference of distances from two fixed points called foci.

Suppose, the foci are situated on the axis Ox symmetrically with respect to the origin (Fig.69), $F_1(c,0)$, $F_2(-c,0)$ and

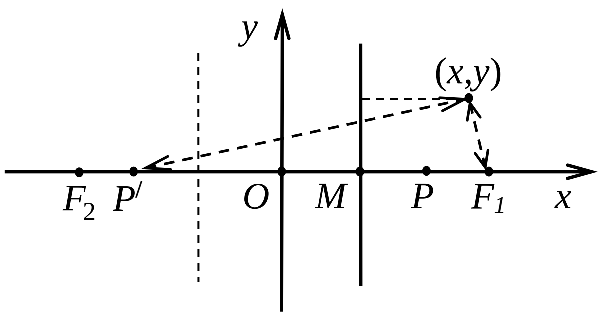


Figure 69

$$|r_1 - r_2| = 2a,$$

where

$$r_1 = \sqrt{(x+c)^2 + y^2},$$

$$r_2 = \sqrt{(x-c)^2 + y^2}.$$

Squaring and expanding both sides give

$$4a^2 = (x+c)^2 + y^2 + (x-c)^2 + y^2 - 2\sqrt{\left((x+c)^2 + y^2\right)\left((x-c)^2 + y^2\right)},$$

$$4a^2 = 2x^2 + 2y^2 + 2c^2 - 2\sqrt{\left((x+c)^2 + y^2\right)\left((x-c)^2 + y^2\right)},$$

$$-2a^2 + \left(x^2 + y^2 + c^2\right) = \sqrt{\left((x+c)^2 + y^2\right)\left((x-c)^2 + y^2\right)},$$

$$-2a^2 + \left(x^2 + y^2 + c^2\right) = \sqrt{\left(x^2 + c^2 + y^2 - 2cx\right)\left(x^2 + c^2 + y^2 + 2cx\right)},$$

$$4a^4 + \left(x^2 + y^2 + c^2\right)^2 - 4a^2\left(x^2 + y^2 + c^2\right) = \left(x^2 + y^2 + c^2\right)^2 - 4c^2x^2,$$

$$4a^4 - 4a^2x^2 - 4a^2y^2 - 4a^2c^2 + 4x^2c^2 = 0,$$

$$a^4 - a^2x^2 - a^2y^2 - a^2c^2 + x^2c^2 = 0,$$

$$x^2\left(c^2 - a^2\right) - a^2y^2 = a^2\left(c^2 - a^2\right).$$

Dividing both sides of this equation by the right-hand member gives

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

Note. From Fig. 69 It follows that for any triangle with vertices F_1, F_2 and any point of hyperbola we have

$$\begin{cases} r_1 < r_2 + 2c \\ r_2 < r_1 + 2c \end{cases} \Leftrightarrow \begin{cases} r_1 - r_2 < 2c \\ r_2 - r_1 < 2c \end{cases} \Leftrightarrow |r_1 - r_2| = 2a < 2c.$$

Thus, $c>a$ and therefore $c^2 - a^2 > 0$ and the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = c^2 - a^2$. This equation is called *the canonical equation of hyperbola*.

Note 1. The equation of hyperbola could be found directly from initial definition of hyperbola as we got it for ellipse. Since for hyperbola $\varepsilon > 1$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-\varepsilon^2)} = 1,$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(\varepsilon^2 - 1)} = 1,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2(\varepsilon^2 - 1)$.

Note 2. From the formulas obtained above, namely

$$c = a\varepsilon,\ d_0 = \frac{a}{\varepsilon},$$

it follows that directrices are situated closer to the origin than foci.

Equations of the directrices are

$$x = \pm\frac{a}{\varepsilon},$$

where

$$\varepsilon = \frac{c}{a}.$$

Properties of hyperbola graph:

1) Axes O*x* and O*y* are axes of hyperbola symmetry. The origin $(0,0)$ is *a center* of hyperbola.

2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Leftrightarrow \frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \Rightarrow \frac{x^2}{a^2} \geq 1 \Rightarrow |x| > |a|$, i.e. the graph of hyperbola is situated outside the band of width 2*a* unbounded in vertical direction.

3) If $y = 0$ then $x = \pm a$. Obtained points $V_1(a,0), V_2(-a,0)$ of intersection with the axis O*x* are called *the vertices* of this hyperbola. The nomenclature of the

hyperbola is slightly different from that of an ellipse. *The transverse axis* is of length 2*a* and is the distance between the vertices of the hyperbola. *The conjugate axis* is of length 2*b* and is perpendicular to the transverse axis.

4) For any point of hyperbola from the right semi-plane (x>0) we have

$$y_h = \pm b\sqrt{\frac{x^2}{a^2}-1} = \pm\frac{b}{a}x \pm b\sqrt{\frac{x^2}{a^2}-1} \mp \frac{b}{a}x =$$

$$= \pm\frac{b}{a}x \pm \left(b\sqrt{\frac{x^2}{a^2}-1} - \frac{b}{a}x \right) = \pm\frac{b}{a}x \pm \frac{b}{a}\left(\sqrt{x^2-a^2} - x\right) =$$

$$= \pm\frac{b}{a}x \pm \frac{b}{a}\left(\frac{x^2-a^2-x^2}{\sqrt{x^2-a^2}+x} \right) = \pm\frac{b}{a}x \pm \frac{b}{a}\left(\frac{-a^2}{\sqrt{x^2-a^2}+x} \right).$$

Thus

$$y_h - \left(\pm\frac{b}{a}\right) = \frac{b}{a}\left(\frac{-a^2}{\sqrt{x^2-a^2}+x} \right) \xrightarrow[x\to+\infty]{} 0.$$

It means that while $x \to +\infty$ the branch of hyperbola tends (becomes extremely close) to the straight lines $y = \pm\frac{b}{a}x$.

These straight lines

$$y = \pm\frac{b}{a}x$$

are called *the asymptotes* of the hyperbola.

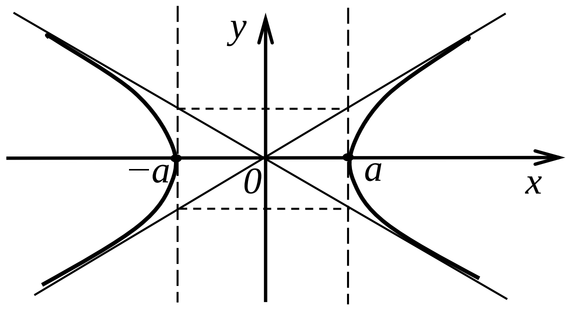


Figure 70

Result of the analysis made above is presented on Fig.70.

Whenever the foci are on the Oy axis and the directrices are straight lines of the form $y = \pm d_0$, the equation of the hyperbola takes form

$$-\frac{x^2}{b^2(\varepsilon^2-1)} + \frac{y^2}{b^2} = 1,$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $F_1(0,c) = F_1(0,b\varepsilon)$, $F_2(0,-c) = F_2(0,-b\varepsilon)$, $|r_1 - r_2| = 2b$, $a^2 = b^2(\varepsilon^2-1)$.

This equation represents a hyperbola with its transverse axis on the axis Oy called *the Conjugate Hyperbola*.

Note 1. Graph of the conjugate hyperbola possesses the same properties as that of common hyperbola, except properties 2) and 3). Vertices of the conjugate hyperbola are points $V_1(0,b),\ V_2(0,-b)$. Moreover,

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1+\frac{x^2}{a^2}\ge 1$$

$$\Rightarrow |y|>b.$$

Graph of conjugate hyperbola is presented by Fig. 71.

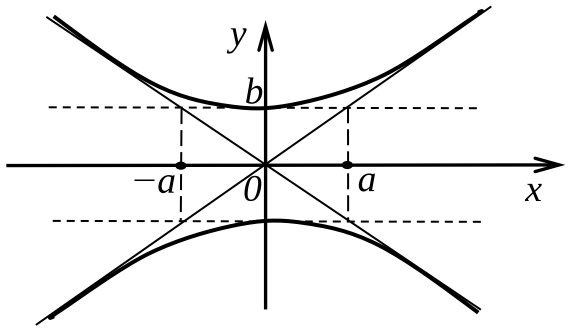


Figure 71

Note 2. If the center of hyperbola is in the point (x_0, y_0) but hyperbola axes are parallel to coordinate axes, then the canonical equation of this hyperbola has the following form

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$

or

$$-\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1.$$

Example. Plot the graph of the following hyperbola:

$$x^2-4x-4y^2-8y+36=0.$$

To plot this hyperbola we should first reduce its equation to the canonical form. Let us complete the squares in x and y in the equation:

$$x^2-4x-4y^2-8y-36=x^2-4x-4\left(y^2+2y\right)+36=$$

$$=x^2-4x+4-4-4\left(y^2+2y+1-1\right)+36=$$

$$=\left(x-2\right)^2-4-4\left(y+1\right)^2+4+36=$$

$$=\left(x-2\right)^2-4\left(y+1\right)^2+36=0.$$

After transposing free member 36 to the right-hand side and dividing the equation by it we get:

$$-\frac{(x-2)^2}{36}+\frac{(y+1)^2}{9}=1.$$

That is a conjugate hyperbola with semi-axes $a=\sqrt{36}=6$, $b=\sqrt{9}=3$ and the vertex $(2,-1)$. Its graph is presented on Fig.72.

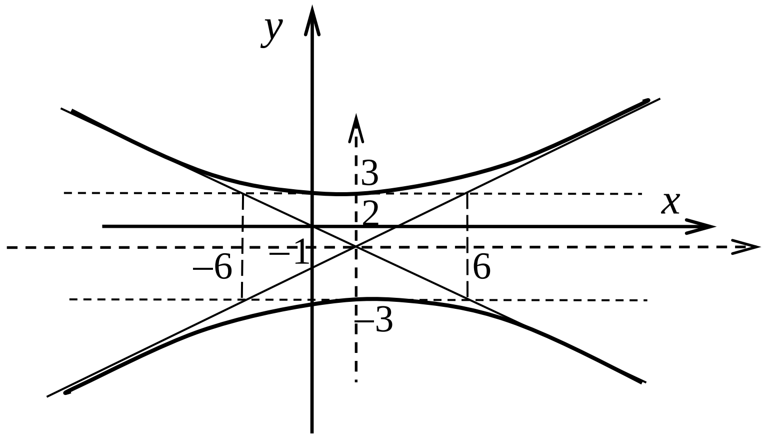


Figure 72

2.4.6. Transformation of Cartesian Coordinates in Plane

It is sometimes desirable in the discussion of problems to change the origin and axes of coordinates by either changing the origin without changing the direction of the axes or changing the direction of the axes without changing the origin or changing both origin and axes. Each of these processes is called a transformation of coordinates.

Transformation 1: a parallel shift (Fig.73), i.e. the transformation when the origin is shifted in other point (x_0, y_0) but axes save their direction. In this case

$$(x,y)=\overrightarrow{OP}=\overrightarrow{OO'}+\overrightarrow{O'P}=$$
$$=(x_0,y_0)+(x',y')=(x_0+x',\ y_0+y').$$

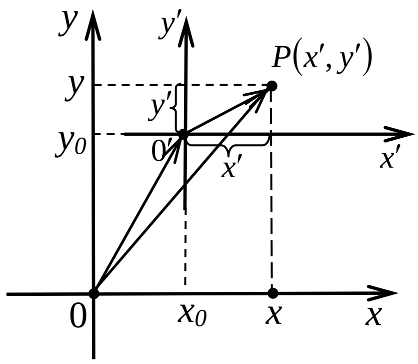


Figure 73

Therefore the dependence between old coordinates (x,y) of the point P and new ones (x',y') is

$$x=x'+x_0,\quad y=y'+y_0.$$

Transformation 2: a turn (Fig. 74), i.e. the transformation when coordinate axes are turned on some angle φ anticlockwise but the origin is saved. At that the clockwise turn is made with negative angle φ.

Suppose, that point $P(x, y)$ has the following polar coordinates in plane Oxy:

$$(\rho, \theta), \text{ i.e. } \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

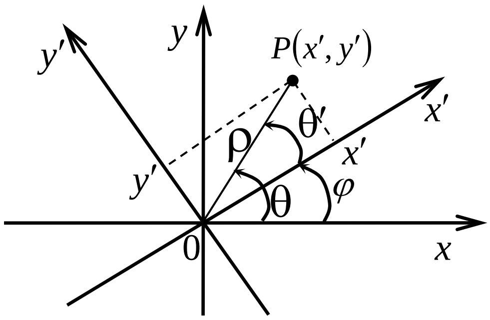


Figure 74

Then in plane $Ox'y'$ obtained by turn of the coordinate axes this point has coordinates:

$$(\rho, \theta'), \text{ i.e. } \begin{cases} x' = \rho\cos\theta' \\ y' = \rho\sin\theta' \end{cases}$$

where $\theta = \varphi + \theta'$.

Therefore

$$x = \rho\cos\theta = \rho\cos(\varphi + \theta') = \underbrace{\rho\cos\theta'}_{x'}\cos\varphi - \underbrace{\rho\sin\theta'}_{y'}\sin\varphi = x'\cos\varphi - y'\sin\varphi,$$

$$y = \rho\sin\theta = \rho\sin(\varphi + \theta') = \underbrace{\rho\cos\theta'}_{x'}\sin\varphi + \underbrace{\rho\sin\theta'}_{y'}\cos\varphi = x'\sin\varphi + y'\cos\varphi.$$

So, the dependence between old coordinates (x, y) of the point P and new ones (x', y') is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

where the transformation matrix is called the turn matrix.

The third type of transformation is a combination of the given above two types.

Note. Except these three types of transformation we can consider other ones, but they either change the relative orientation of the axes or do not save the distance. We are not interested in such transformations for our further discussions, so they are skipped here.

2.4.7. Reducing the General Equation of the Second Order Curve to a Canonical Form

Suppose, we have some general equation of the second order curve

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

The question is: what curve does this equation present? To answer this question we should reduce this equation to a canonical form.

Theorem 1. By turn of coordinate system the general equation of the second order curve with coefficient $B \neq 0$ can be reduced to the form

$$A_1 x'^2 + C_1 y'^2 + 2D_1 x' + 2E_1 y' + F = 0.$$

Proof. Since

$$x = x'\cos\varphi - y'\sin\varphi, \quad y = x'\sin\varphi + y'\cos\varphi,$$

we have

$$A\left(x'^2\cos^2\varphi + y'^2\sin^2\varphi - 2x'y'\cos\varphi\sin\varphi\right) +$$
$$+ C\left(x'^2\sin^2\varphi + y'^2\cos^2\varphi + 2x'y'\cos\varphi\sin\varphi\right) +$$
$$+ 2B\left(x'^2\cos\varphi\sin\varphi - y'^2\cos\varphi\sin\varphi + x'y\left(\cos^2\varphi - \sin^2\varphi\right)\right) +$$
$$+ 2Dx'\cos\varphi - 2Dy'\sin\varphi + 2Ex'\sin\varphi + 2Ey'\cos\varphi + F = 0.$$

After factoring out we get

$$x'^2\left(A\cos^2\varphi + C\sin^2\varphi + 2B\cos\varphi\sin\varphi\right) + y'^2\left(A\sin^2\varphi + C\cos^2\varphi - 2B\cos\varphi\sin\varphi\right) +$$
$$+ x'y'\left(-2A\cos\varphi\sin\varphi + 2C\cos\varphi\sin\varphi + 2B\left(\cos^2\varphi - \sin^2\varphi\right)\right) +$$
$$+2x'\left(D\cos\varphi + E\sin\varphi\right) + 2y'\left(E\cos\varphi - D\sin\varphi\right) + F = 0.$$

Vanishing the coefficient at $x'y'$ we obtain

$$-2A\cos\varphi\sin\varphi + 2C\cos\varphi\sin\varphi + 2B\left(\cos^2\varphi - \sin^2\varphi\right) = 0,$$
$$(C - A)\sin 2\varphi + 2B\cos 2\varphi = 0,$$
$$\boxed{\cot 2\varphi = \frac{A - C}{2B}}.$$

The asked transformation can be therefore made by any angle φ, satisfying the last equation. After this the equation under consideration takes form

$$A'x'^2 + C'y'^2 + 2D'x' + 2E'y' + F = 0,$$

where

$A' = A\cos^2\varphi + C\sin^2\varphi + 2B\cos\varphi\sin\varphi$, $C' = A\sin^2\varphi + C\cos^2\varphi - 2B\cos\varphi\sin\varphi$,
$D' = D\cos\varphi + E\sin\varphi$, $E' = E\cos\varphi - D\sin\varphi$.

Theorem is proven.

Note. Since

$$\cot 2\varphi = \frac{1-\tan^2\varphi}{2\tan\varphi},$$

the value of the angle φ can be found also from equation:

$$B\tan^2\varphi + (A-C)\tan\varphi - B = 0,$$

while

$$\cos\varphi = \frac{1}{\sqrt{1+\tan^2\varphi}}, \quad \sin\varphi = \tan\varphi\cos\varphi.$$

Theorem 2. By parallel shift the equation $Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0$ of the second order curve with $A \neq 0$, $C \neq 0$ can be reduced to the form $A(x')^2 + C\left(y'^2\right) + F_1 = 0$.

Proof. Let us consider the parallel shift

$$x = x' + x_0, \ y = y' + y_0.$$

Then considered equation can be rewritten in the following form

$$Ax'^2 + 2Ax'x_0 + Ax_0^2 + Cy'^2 + 2Cy'y_0 + Cy_0^2 + 2Dx' + 2Dx_0 + 2Ey' + 2Ey_0 + F =$$

$$= Ax'^2 + Cy'^2 + x'(2Ax_0 + 2D) + y'(2Cy_0 + 2E) + Ax_0^2 + Cy_0^2 + 2Dx_0 + 2Ey_0 + F.$$

Vanishing the coefficients at x', y' we get

$$x_0 = -\frac{D}{A}, \ y_0 = -\frac{E}{C}$$

and equation

$$Ax'^2 + Cy'^2 + F_1 = 0,$$

where $F_1 = Ax_0^2 + Cy_0^2 + 2Dx_0 + 2Ey_0 + F$. ***Theorem is proven.***

Note 1. Values x_0, y_0 for parallel shift can be found also by the procedure of completing squares in the given equation.

Note 2. If one of coefficients, either A or C, is equal to zero then parallel shift should be made only along the axis corresponding to nonzero coefficient.

Theorem 3. By parallel shift the equations $Ax^2+Ey+F=0$ and $Cy^2+Dx+F=0$, where $E\cdot F\neq 0$, $D\cdot F\neq 0$, can be reduced to the form

$$Ax^2+E'y=0,\ Cy^2+Dx'=0,\ \text{relatively.}$$

Proof. Let us consider the parallel shift

$$x=x',\ y=y'+y_0.$$

Then the first equation can be rewritten in the following form

$$Ax'^2+Ey'+Ey_0+F=0.$$

Vanishing free coefficient of this equation we get

$$y_0=-\frac{F}{E}.$$

In the same way by means of the parallel shift

$$x=x'-\frac{F}{D},\ y=y'$$

the second equation can be reduced to the required form. ***Theorem is proven.***

From Theorems 1-3 It follows that the general equation of the second order curve can be reduced to the one of the following forms:

(1): $Ax^2+Cy^2+F=0$, where $A\cdot C\neq 0$.

If $F\neq 0$ then it can be an equation of

– ellipse ($A\cdot C>0,\ A\cdot F<0$): $\dfrac{x^2}{-\frac{F}{A}}+\dfrac{y^2}{-\frac{F}{C}}=1$;

– imaginary ellipse ($A\cdot C>0,\ A\cdot F>0$): $\dfrac{x^2}{\frac{F}{A}}+\dfrac{y^2}{\frac{F}{C}}=-1$;

– hyperbola ($A\cdot C<0,\ A\cdot F<0$): $\dfrac{x^2}{-\frac{F}{A}}-\dfrac{y^2}{\frac{F}{C}}=1$;

– conjugate hyperbola ($A\cdot C<0,\ A\cdot F>0$): $-\dfrac{x^2}{\frac{F}{A}}+\dfrac{y^2}{-\frac{F}{C}}=1$.

If $F=0$ then it can be an equation of

– two crossing straight lines ($A\cdot C<0$): $y=\pm x\sqrt{\dfrac{-A}{C}}$;

– a point ($A\cdot C>0$): $x=0, y=0$.

(2): $Ax^2+Ey=0$, where $A\neq 0$.

If $E\neq 0$ then it is an equation of the parabola with symmetry axis O*x*: $x^2=-\dfrac{E}{A}y=2py$.

If $E=0$ then it is an equation of the axis O*y*: $x=0$.

(3): $Cy^2+Dx=0$, where $C\neq 0$.

If $D\neq 0$ then it is an equation of the parabola with symmetry axis O*y*: $y^2=-\dfrac{D}{A}x=2px$.

If $D=0$ then it is an equation of the axis O*x*: $y=0$.

(4): $Ax^2+F=0$, where $A\neq 0$.

If $F\neq 0$ then that is an equation of

– two straight lines parallel to axis O*y* ($A\cdot F<0$): $x=\pm\sqrt{\dfrac{-F}{A}}$;

– two imaginary straight lines ($A\cdot F>0$): $x^2=\dfrac{-F}{A}$.

If $F=0$ then it is an equation of the axis O*y*: $x=0$.

(5): $Cy^2+F=0$, where $C\neq 0$.

If $F\neq 0$ then that is an equation of

– two straight lines parallel to axis O*x* ($C\cdot F<0$): $y=\pm\sqrt{\dfrac{-F}{C}}$;

– two imaginary straight lines ($C\cdot F>0$): $y^2=\dfrac{-F}{C}$.

If $F=0$ then it is an equation of the axis O*x*: $y=0$.

Example. The following equation is given

$$25x^2-14xy+25y^2+64x-64y-224=0.$$

Reduce it to a canonical form and plot a graph.

$$\cot 2\varphi = \frac{A-C}{B} = \frac{25-25}{-14} = 0 \Rightarrow \varphi = \frac{\pi}{4}.$$

Therefore

$$x = x_1 \cos\varphi - y_1 \sin\varphi = \frac{x_1 - y_1}{\sqrt{2}},$$

$$y = x_1 \sin\varphi + y_1 \cos\varphi = \frac{x_1 + y_1}{\sqrt{2}}.$$

The substitution of these equalities into the given equation and simplification give

$$18x_1^2 + 32y_1^2 - 64\sqrt{2}y_1 - 224 = 0.$$

After parallel shift

$$x_1 = x_2, y_1 = y_2 + \sqrt{2}$$

we get

$$18x_2^2 + 32y_2^2 - 288 = 0.$$

This equation is the equation of the ellipse with canonical form

$$\frac{x_2^2}{16} + \frac{y_2^2}{9} = 1.$$

Here $a = 4, b = 3$. Graph of this ellipse is presented on the Fig.75.

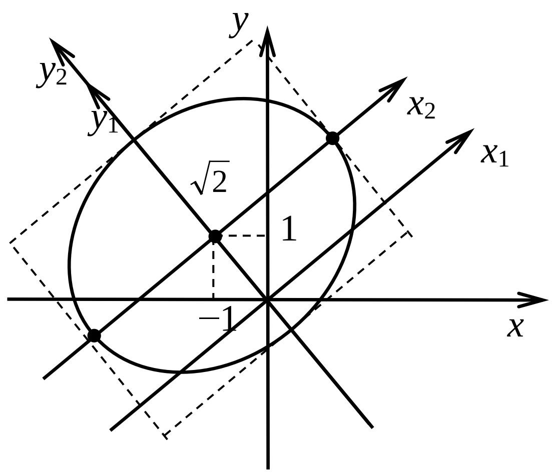


Figure 75

2.4.8. Invariance of the Expressions $AC - B^2$ and $A + C$ at Turn of Coordinate Axes. Classification of the Second Order Curves

It is appeared that the turn made in the Theorem 1 saves the value of the expression $AC - B^2$, i.e.

$$AC - B^2 = \tilde{A}\tilde{C} - \tilde{B}^2 = \tilde{A}\tilde{C}.$$

Indeed,

$$\tilde{A} = A\cos^2\varphi + 2B\cos\varphi\sin\varphi + C\sin^2\varphi =$$

$$= \frac{1}{2}\left(A + A\cos 2\varphi + 2B\sin 2\varphi + C - C\cos 2\varphi\right),$$

$$\tilde{C} = A\sin^2\varphi + 2B\sin\varphi\cos\varphi + C\cos^2\varphi =$$

$$\frac{1}{2}\left(A - A\cos 2\varphi - 2B\sin 2\varphi + C + C\cos 2\varphi\right).$$

Therefore,

$$\tilde{A} = \frac{1}{2}(A + C) + \frac{1}{2}(A - C)\cos 2\varphi + B\sin 2\varphi =$$

$$A = \frac{1}{2}(A + C) + B\sin 2\varphi\left(\frac{A - C}{B}\cos 2\varphi + 1\right) =$$

$$= \frac{1}{2}(A + C) + B\sin 2\varphi\left(1 + \cot^2 2\varphi\right) = \frac{1}{2}(A + C) + \frac{B\sin 2\varphi}{\sin^2 2\varphi} = \frac{A + C}{2} + \frac{B}{\sin 2\varphi}.$$

In the similar way

$$\tilde{C} = \frac{1}{2}(A + C) - \frac{1}{2}(A - C)\cos 2\varphi - B\sin 2\varphi = \frac{A + C}{2} - \frac{B}{\sin 2\varphi}.$$

So,

$$\tilde{A}\tilde{C} = \frac{(A + C)^2}{4} - \frac{B^2}{\sin^2 2\varphi}.$$

Since

$$\frac{1}{\sin^2 2\varphi} = 1 + \cot^2 2\varphi = 1 + \frac{(A - C)^2}{4B^2},$$

we finally get

$$\tilde{A}\tilde{C} = \frac{(A + C)^2}{4} - B^2 - \frac{(A - C)^2}{4} = AC - B^2.$$

At the same time

$$\tilde{A} + \tilde{C} = \frac{A + C}{2} + \frac{B}{\sin 2\varphi} + \frac{A + C}{2} - \frac{B}{\sin 2\varphi} = A + C.$$

The property of invariance of the expression $AC - B^2$ gives the opportunity to classify the second order curves in the following way.

Classification of the Second Order Curves:

$AC - B^2 > 0 \Rightarrow$ the curve is of elliptic kind (ellipse, imaginary ellipse, point);

$AC - B^2 < 0 \Rightarrow$ the curve is of hyperbolic kind (hyperbola, conjugate hyperbola, two crossing S.L.);

$AC - B^2 = 0 \Rightarrow$ the curve is of parabolic kind (parabola, two parallel or imaginary straight lines).

2.4.9. Linear Operator. Matrix of Linear Operator

Definition. Suppose M and L are linear spaces. Map $\mathbf{A}: L \to M$, establishing to any element x of the space L some element *y* of the space *M*, is called an operator $\mathbf{A}$, operating from L to M.

The action of operator can be written as $y = \mathbf{A}x$.

Definition. Operator $\mathbf{A}$ is called linear if for any two elements $x_1, x_2 \in L$ and any numbers $\alpha, \beta \in R$ the following is valid

$$\mathbf{A}(\alpha x_1 + \beta x_2) = \alpha \mathbf{A} x_1 + \beta \mathbf{A} x_2,$$

i.e. it satisfies the properties of additivity and homogeneity.

Definition. Linear operator $\mathbf{A}$ operating from the linear space L to itself is called linear transformation of the linear space L.

Example. Suppose,

operator $\mathbf{A}_1$ is the operator of the anticlockwise rotation of any vector from R^2 on angle φ; operator $\mathbf{A}_2$ puts in correspondence to any vector of, for example, R^3 its scalar product with some fixed vector, i.e. $\mathbf{A}_2 \overline{x} = (\overline{x}, \overline{a})$;

operator $\mathbf{A}_3$ puts in correspondence to any vector of R^3 its vector product with some fixed vector, i.e. $\mathbf{A}_3 \overline{x} = [\overline{x}, \overline{b}]$;

operator $\mathbf{A}_4$ puts in correspondence to any vector its first coordinate in some fixed basis, i.e. $\mathbf{A}_4 \overline{x} = x_1$;

operator $\mathbf{A}_5$ puts in correspondence to any vector its module squared, i.e. $\mathbf{A}_5 \overline{x} = |\overline{x}|^2$.

It could be shown that the operators $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are linear, but $\mathbf{A}_5$ is not linear. Moreover, only $\mathbf{A}_1$ and $\mathbf{A}_3$ are linear transformations of the corresponding linear vector spaces.

Basic Properties and Types of Linear Operators:

1) $(\mathbf{A}+\mathbf{B})x = \mathbf{A}x + \mathbf{B}x$;

2) $(\lambda \mathbf{A})x = \lambda(\mathbf{A}x) \quad \lambda \in R$;

3) $\mathbf{O}x = 0 \ \ \forall x \in L$, i.e. $\mathbf{O}$ is a null or zero operator;

4) $\mathbf{I}x = x \ \ \forall x \in L$, i.e. $\mathbf{I}$ is an identical or unit operator;

5) $(-\mathbf{A})x = -(\mathbf{A}x)$, i.e. there is an opposite or negative operator $-\mathbf{A}$ to any operator $\mathbf{A}$.

Note. The set of all linear operators is a linear space.

If L and M are finite-dimensional, and one has chosen the bases in those spaces, then every linear operator $\mathbf{A}$ operating from L to M can be represented as a *matrix*; that is very convenient since it allows to simplify the concrete calculations. Conversely, matrices yield examples of linear operators: if A is a real m-by-n matrix, then the rule $\boldsymbol{A}(x)=Ax$ describes a linear operator $R^n \to R^m$.

Suppose L is n-dimensional space, M is m-dimensional space, $e_1, e_2, ..., e_n$ and $e'_1, e'_2, ..., e'_m$ are bases of these spaces relatively. Then every element of these spaces can be uniquely determined by the coordinates in the corresponding bases.

Since $\mathbf{A}e_1, \mathbf{A}e_2, ..., \mathbf{A}e_n \in M$, then

$$\begin{cases} \mathbf{A}e_1 = a_{11}e'_1 + a_{21}e'_2 + ... + a_{m1}e'_m; \\ \mathbf{A}e_2 = a_{12}e'_1 + a_{22}e'_2 + ... + a_{m2}e'_m; \\ \qquad \cdots \\ \mathbf{A}e_n = a_{1n}e'_1 + a_{2n}e'_2 + ... + a_{mn}e'_m. \end{cases}$$

Suppose $\mathbf{A}x = y$. Then

$$x = \sum_{i=1}^{n} x_i e_i \,, \quad y = \sum_{k=1}^{m} y_k e'_k \,,$$

$$\mathbf{A}x = \sum_{i=1}^{n} x_i \mathbf{A}e_i = \sum_{i=1}^{n} x_i \left(\sum_{k=1}^{m} a_{ki} e'_k \right) = \sum_{k=1}^{m} e'_k \left(\sum_{i=1}^{n} x_i a_{ki} \right) = \sum_{k=1}^{m} y_k e'_k ,$$

and therefore,

$$y_k = \sum_{i=1}^{n} a_{ki} x_i .$$

The obtained result means that the action of the operator $\mathbf{A}$ can be replaced by the operation of multiplication of the matrix $A = (a_{ki})_{k,i}$ by x, i.e.

$$y = \mathbf{A}x = Ax = \begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ \dots & \dots & \dots \dots \\ a_{m1} & a_{m2} & \dots a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}.$$

Note 1. The k-th column of the constructed matrix A is decomposition of the $\mathbf{A}e_k$ in the basis of the space M.

Note 2. Matrix of the linear transformation $\mathbf{A}$ from L to M is square matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ \dots & \dots & \dots \dots \\ a_{n1} & a_{n2} & \dots a_{nn} \end{pmatrix}$$

Note 3. A single linear operator may be represented by many matrices. This is because the values of the elements of the matrix depend on the bases that are chosen.

Note 4. It could be shown that for linear transformation $\mathbf{A}$ its matrix in the new basis can be found by formula

$$\tilde{A} = B^{-1}AB,$$

where B is the matrix of transition from old basis to new one. As it was obtained in Section 2.1.6 the columns of this matrix are coordinates of the new basis in the old one.

Note 5. Since for identical operator we have: $\mathbf{I}e_i = e_i$, the matrix of the identical operator is unit matrix.

For null operator $\mathbf{O}e_i = 0$, thus the matrix of this operator is zero matrix.

Since $(\mathbf{A} + \mathbf{B})e_i = \mathbf{A}e_i + \mathbf{B}e_i$, the matrix of the sum of operators is sum of corresponding matrices.

And finally, since $(\alpha\mathbf{A})e_i = \mathbf{A}(\alpha e_i) = A(\alpha e_i) = (\alpha A)e_i$, the matrix of $\alpha\mathbf{A}$ is αA. Thus, matrix of linear combination of the operators is the linear combination of their matrices.

Example. Matrices of some special cases of linear transformations of two-dimensional space R^2 are given below:

a) reflection against the origin (Fig. 76.*a*):

$$\text{Since } \begin{cases} \mathbf{A}\bar{i} = -\bar{i} + 0 \cdot \bar{j} \\ \mathbf{A}\bar{j} = 0 \cdot \bar{i} - \bar{j} \end{cases}, \text{ then } A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) reflection against the *x* axis (Fig. 76.*b*):

$$\text{Since } \begin{cases} \mathbf{A}\bar{i} = \bar{i} + 0 \cdot \bar{j} \\ \mathbf{A}\bar{j} = 0 \cdot \bar{i} - \bar{j} \end{cases}, \text{ then } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

c) rotation by 90 degrees anticlockwise (Fig. 76.*c*):

$$\text{Since } \begin{cases} \mathbf{A}\bar{i} = 0 \cdot \bar{i} + \bar{j} \\ \mathbf{A}\bar{j} = -\bar{i} + 0 \cdot \bar{j} \end{cases}, \text{ then } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

d) rotation by angle φ anticlockwise (Fig. 76.*d*):

$$\text{Since } \begin{cases} \mathbf{A}\bar{i} = \cos\varphi\bar{i} + \sin\varphi\bar{j} \\ \mathbf{A}\bar{j} = -\sin\varphi\bar{i} + \cos\varphi\bar{j} \end{cases}, \text{ then } A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

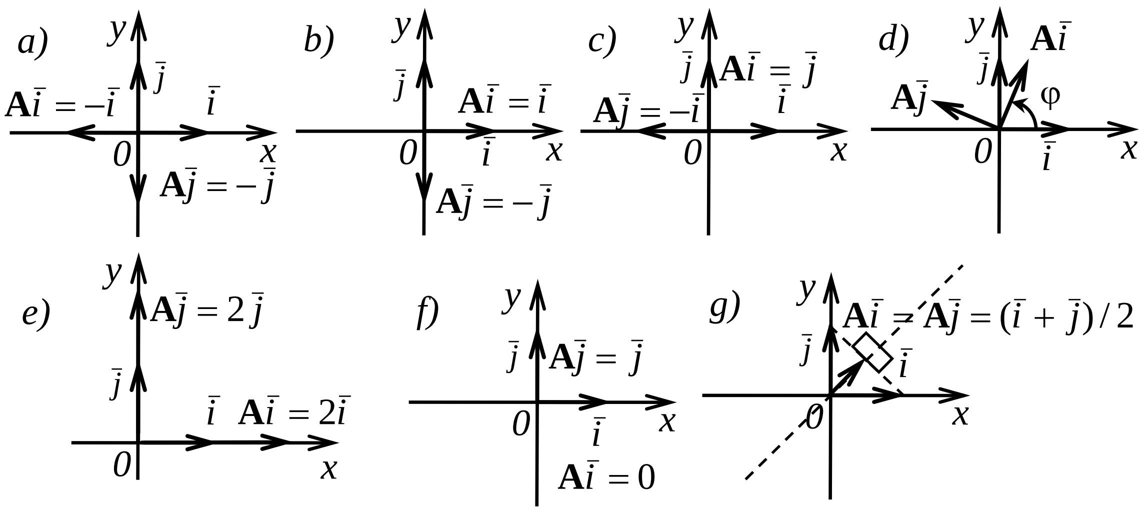


Figure 76

e) scaling by 2 in all directions (Fig. 76.*e*):

$$\text{Since } \begin{cases} \mathbf{A}\bar{i} = 2\bar{i} + 0\cdot\bar{j} \\ \mathbf{A}\bar{j} = 0\cdot\bar{i} + 2\bar{j} \end{cases}, \text{ then } A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

f) projection onto the *Oy* axis (Fig. 76.*f*):

$$\text{Since } \begin{cases} \mathbf{A}\bar{i} = 0\cdot\bar{i} + 0\cdot\bar{j} \\ \mathbf{A}\bar{j} = 0\cdot\bar{i} + \bar{j} \end{cases}, \text{ then } A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

g) projection onto straight line *y*=*x* (Fig. 76.*g*):

$$\text{Since } \begin{cases} \mathbf{A}\bar{i} = 0.5\bar{i} + 0.5\bar{j} \\ \mathbf{A}\bar{j} = 0.5\bar{i} + 0.5\bar{j} \end{cases}, \text{ then } A = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

Note, that

$$\mathbf{A}0 = \mathbf{A}(x - x) = \mathbf{A}x - \mathbf{A}x = 0 \text{ for any operator.}$$

The question is: could $\mathbf{A}x$ be zero-element if *x* is not zero?

Definition. Linear operator is called singular if there is non-zero element $x \in L$ such that $\mathbf{A}x$ is zero, i.e. $\exists x \neq 0 : \mathbf{A}x = 0$. In other case this operator is called non-singular.

Note. For linear transformation to be singular means that its matrix is singular matrix, i.e. $\det A = 0$. Non-singular matrices correspond to non-singular linear transformations.

From the previous example we get: operators from cases *a*)-*e*) are non-singular, but others are singular.

2.4.10. Eigenvalues and Eigenvectors of Linear Operator

Definition. Number λ is called the eigenvalue of the linear operator $\mathbf{A} : L \to L$, if there is non-zero element *x* such that $\mathbf{A}x = \lambda x$. In that case the element *x* is called the eigenvector of the operator $\mathbf{A}$.

Note 1. Eigenvalue λ and eigenvector *x* are also called eigenvalue and eigenvector of the matrix *A* of the operator $\mathbf{A}$.

Note 2. The eigenvector is defined accurate to a constant factor, i.e. if x is eigenvector then kx is eigenvector corresponding to the same eigenvalue, too. Indeed,

$$\mathbf{A}x = \lambda x \Leftrightarrow k(\mathbf{A}x) = k(\lambda x) \Leftrightarrow \mathbf{A}(kx) = \lambda(kx).$$

Theorem. Suppose L is finite-dimensional linear space. Then for the value λ to be eigenvalue of the linear operator $\mathbf{A}: L \to L$ it is necessary and sufficient to be a root of the characteristic equation $\det(A - \lambda E) = 0$, where A is a matrix of $\mathbf{A}$ in some fixed basis.

Proof. Suppose λ is the eigenvalue and x is corresponding eigenvector. Since

$$\mathbf{A}x = \lambda x = \lambda \mathbf{I}x$$

we have

$$(\mathbf{A} - \lambda \mathbf{I})x = 0 \text{ and } x \neq 0$$

or

$$(A - \lambda E)x = 0 \text{ and } x \neq 0.$$

It means that $A - \lambda E$ is singular matrix and $\det(A - \lambda \mathrm{E}) = 0$.

The sufficiency can be proved if one repeats all obtained above conclusions in reverse order. ***Theorem is proven.***

Example. Find eigenvalues and eigenvectors of the matrix

$$\mathrm{A} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let us find the roots of the characteristic equation:

$$|\mathrm{A} - \lambda \mathrm{E}| = \begin{vmatrix} -1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)(2-\lambda) = 0.$$

So, the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 2$.

To find eigenvectors we should solve the equation $(A - \lambda E)x = 0$ for different values of λ.

1) $\lambda_1 = -1$.

$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_2 = 0 \\ 4x_2 + x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - \forall \\ x_2 = 0 \\ x_3 = 0 \end{cases}$. Thus, $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ can be taken as an eigenvector.

2) $\lambda_2 = 3$.

$$\begin{pmatrix} -4 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -4x_1 + 2x_2 = 0 \\ x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - \forall \\ x_2 = 2x_1 \\ x_3 = 0 \end{cases}. \text{ Thus, } X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

3) $\lambda_3 = 2$.

$$\begin{pmatrix} -3 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -3x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_2/3 \\ x_2 - \forall \\ x_3 = -x_2 \end{cases}. \text{ Thus, } X_3 = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}.$$

There are several important theorems about eigenvalues and eigenvectors from the theory of Linear Operators:

Theorem 1. Suppose that matrix of the linear operator from the *n*-dimensional space *L* to *L* is symmetrical. Then this operator has exactly *n* real eigenvalues.

Operator with symmetrical matrix is called symmetric operator.

Theorem 2. Eigenvectors of the symmetric operator corresponding to all different eigenvalues are linearly independent and mutually orthogonal.

Note. It is assumed in the previous theorem that the relation of the orthogonality is defined in *L*. It is defined, for example, in any space with scalar product. In the *n*-dimensional space with orthonormal basis the scalar product can be defined as

$$\left(\vec{a}, \vec{b}\right) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n .$$

Theorem 3. Suppose operator $\mathbf{A}$ is symmetric. Then there are exactly *p* linearly independent and mutually orthogonal eigenvectors corresponding to the multiple eigenvalue λ of the order *p*.

Example. Show that the eigenvalues of the symmetric operator from R^2 to R^2 are different.

Matrix of this operator has a form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then

$$|\mathrm{A} - \lambda \mathrm{E}| = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

and

$$\lambda_{1,2} = \frac{a + c \pm \sqrt{(a+c)^2 - 4ac + 4b^2}}{2} = \frac{a + c \pm \sqrt{(a-c)^2 + 4b^2}}{2},$$

i.e. we obtained two different real values.

Note. It follows from the Theorem 2 that the symmetrical matrix of the second order has two orthogonal eigenvectors corresponding to the different eigenvalues.

2.4.11. Quadratic Form. Reducing the Quadratic Form to a Canonical Form

Definition. Homogeneous polynomial of degree two in a number of variables

$$Q(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

is called the quadratic form of these variables.

Example. Suppose

$$Q_1(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2},\ Q_2(x, y, z) = x^2 - 4xy + z^2,\ Q_3(x, y, z) = zx^2 + xy + 3.$$

Q_1 and Q_2 are quadratic forms, but Q_3 is not quadratic form since the first term of Q_3 is cubic monomial and the last term is a constant.

Note. If a quadratic form is given then all like terms are already collected. It means that we have, for example, one term Axy but not two $a_{12}x_1x_2$ and $a_{21}x_2x_1$. Thus, we always assume, that $a_{12} = a_{21} = A/2$.

So, for the given in example quadratic form Q_2 we have: $a_{12} = a_{21} = -4/2 = -2$.

In general we assume that

$$a_{ij} = a_{ji} \quad \forall i, j = 1,2...n.$$

Definition. Symmetrical matrix $A=\left(a_{ij}\right)$ of monomial coefficients in quadratic form is called the matrix of this quadratic form.

By means of this matrix the quadratic form Q can be expressed in terms of matrices as

$$Q(\mathrm{X})=Q\left(x_1,x_2,...,x_n\right)=\left(x_1 \quad x_2 \quad \dots \quad x_n\right)\begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}=X^T AX .$$

Definition. Standard (or canonical) form of the quadratic form is

$$Q\left(x_1,x_2,...,x_n\right)=\sum_{i=1}^{n} a_{ii}x_i^2 .$$

Note 1. It follows from definition that the matrix of the quadratic form in canonical form is diagonal matrix.

Note 2. It is obvious that by means of linear change of variables we can find another form of the quadratic form. And the natural question that arises in this situation is how to find such a change of variable to get more simple canonical form?

Theorem. Quadratic form considered in the basis of orthogonal and normalized (of unit length) eigenvectors of the symmetrical matrix corresponding to this quadratic form has a canonical form. Moreover, coefficients of monomials are eigenvalues of this matrix.

Proof. Suppose we consider new basis $\tilde{e}_1,\tilde{e}_2,\tilde{e}_3,\dots\tilde{e}_n$ consisting of the set of the unit eigenvectors of A. Then $X=B\tilde{X}$, where $\tilde{X}$ are coordinates of this vector in new basis, B is transition matrix consisting of the coordinates of eigenvectors.

Therefore

$$Q=X^T AX=(B\tilde{X})^T A(B\tilde{X})=\tilde{X}^T B^T AB\tilde{X}=\tilde{X}^T\tilde{A}\tilde{X} ,$$

where $\tilde{A}$ is a matrix of the operator corresponding to the A in the new basis.

$$\tilde{A}=B^T AB=\left(\begin{array}{c|c|c|c} \tilde{e}_1 & \tilde{e}_2 & \dots & \tilde{e}_n \end{array}\right)^T\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\left(\begin{array}{c|c|c|c} \tilde{e}_1 & \tilde{e}_2 & \dots & \tilde{e}_n \end{array}\right)=$$

$$= \begin{pmatrix} \tilde{e}_1^T \\ \hline \cdots \\ \hline \tilde{e}_n^T \end{pmatrix} \left(A\tilde{e}_1 \middle| A\tilde{e}_2 \middle| \ldots \middle| A\tilde{e}_n \right) = \begin{pmatrix} \tilde{e}_1^T \\ \hline \cdots \\ \hline \tilde{e}_n^T \end{pmatrix} \left(\lambda_1\tilde{e}_1 \middle| \lambda_2\tilde{e}_2 \middle| \ldots \middle| \lambda_n\tilde{e}_n \right) =$$

$$= \begin{pmatrix} (\tilde{e}_1, \lambda_1\tilde{e}_1) & (\tilde{e}_1, \lambda_2\tilde{e}_2) & \ldots & (\tilde{e}_1, \lambda_n\tilde{e}_n) \\ (\tilde{e}_2, \lambda_1\tilde{e}_1) & (\tilde{e}_2, \lambda_2\tilde{e}_2) & & (\tilde{e}_2, \lambda_n\tilde{e}_n) \\ \vdots & & \ddots & \\ (\tilde{e}_n, \lambda_1\tilde{e}_1) & (\tilde{e}_n, \lambda_2\tilde{e}_2) & \ldots & (\tilde{e}_n, \lambda_n\tilde{e}_n) \end{pmatrix} = \begin{pmatrix} \lambda_1(\tilde{e}_1, \tilde{e}_1) & \lambda_2(\tilde{e}_1, \tilde{e}_2) & \ldots & \lambda_n(\tilde{e}_1, \tilde{e}_n) \\ \lambda_1(\tilde{e}_2, \tilde{e})_1 & \lambda_2(\tilde{e}_2, \tilde{e}_2) & & \lambda_n(\tilde{e}_2, \tilde{e}_n) \\ \vdots & & \ddots & \\ \lambda_1(\tilde{e}_n, \tilde{e}_1) & \lambda_2(\tilde{e}_n, \tilde{e}_2) & \ldots & \lambda_n(\tilde{e}_n, \tilde{e}_n) \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1 |\tilde{e}_1|^2 & \lambda_2 \cdot 0 & \ldots & \lambda_n \cdot 0 \\ \lambda_1 \cdot 0 & \lambda_2 |\tilde{e}_2|^2 & & \lambda_n \cdot 0 \\ \vdots & & \ddots & \\ \lambda_1 \cdot 0 & \lambda_2 \cdot 0 & \ldots & \lambda_n |\tilde{e}_n|^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \cdot 1 & \lambda_2 \cdot 0 & \ldots & \lambda_n \cdot 0 \\ \lambda_1 \cdot 0 & \lambda_2 \cdot 1 & & \lambda_n \cdot 0 \\ \vdots & & \ddots & \\ \lambda_1 \cdot 0 & \lambda_2 \cdot 0 & \ldots & \lambda_n \cdot 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.$$

Thus, the obtained matrix of quadratic form is diagonal, so we finally have:

$$Q = \lambda_1\tilde{x}_1^2 + \lambda_2\tilde{x}_2^2 + \ldots + \lambda_n\tilde{x}_n^2 .$$

Theorem is proven.

Note 1. Matrix B satisfies the condition

$$B^T B = \begin{pmatrix} \tilde{e}_1^T \\ \hline \cdots \\ \hline \tilde{e}_n^T \end{pmatrix} \left(\tilde{e}_1 \middle| \tilde{e}_2 \middle| \ldots \middle| \tilde{e}_n \right) = \begin{pmatrix} (\tilde{e}_1, \tilde{e}_1) & (\tilde{e}_1, \tilde{e}_2) & \ldots & (\tilde{e}_1, \tilde{e}_n) \\ (\tilde{e}_2, \tilde{e})_1 & (\tilde{e}_2, \tilde{e}_2) & & (\tilde{e}_2, \tilde{e}_n) \\ \vdots & & \ddots & \\ (\tilde{e}_n, \tilde{e}_1) & (\tilde{e}_n, \tilde{e}_2) & \ldots & (\tilde{e}_n, \tilde{e}_n) \end{pmatrix} =$$

$$= \begin{pmatrix} |\tilde{e}_1|^2 & 0 & \ldots & 0 \\ 0 & |\tilde{e}_2|^2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & |\tilde{e}_n|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & 1 \end{pmatrix} = E .$$

In the same way It could be shown that $BB^T = E$. Thus, $B^{-1} = B^T$.

Note 2. Matrix $\tilde{A}$ from the proof of the theorem could be constructed directly from the definition of operator matrix:
Since $\mathbf{A}\tilde{e}_i = \lambda_i \tilde{e}_i = 0 \cdot \tilde{e}_1 + 0 \cdot \tilde{e}_2 + \ldots + \lambda_i \tilde{e}_i + 0 \cdot \tilde{e}_{i+1} + \ldots + 0 \cdot \tilde{e}_n$, then

$$\tilde{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Note 3. It is clear that this technique can be used also to reduce the equation of the second order curve to the canonical form. In this case we consider quadratic form of two variables with symmetrical matrix of the second order. As it was written above It always has two different eigenvalues and thus two orthogonal eigenvectors of unit length. So, if the matrix B constructed from them has positive determinant then the obtained linear transformation of coordinates is just the turn of coordinate system by some angle φ anticlockwise, i.e.

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \text{ and } b_{11} = \cos\varphi,\ b_{21} = \sin\varphi .$$

Example. Reduce the equation of the second order curve to a canonical form:

$$x^2 + y^2 + 4xy + 4x + 2y - 5 = 0 .$$

Quadratic form corresponding to the given equation and its matrix are

$$\mathrm{Q}(x,y) = x^2 + y^2 + 4xy , \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Let us find its eigenvalues and eigenvectors to form the matrix of the turn.

$\lambda : \det(A - \lambda E) = 0$, i.e. $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = (1-\lambda-2)(1-\lambda+2) = 0$;

$(-\lambda - 1)(3 - \lambda) = 0$; $\lambda_1 = -1,\ \ \lambda_2 = 3$.

1) $\lambda_1 = -1:\ 2x_1 + 2x_2 = 0 \Rightarrow X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $|X_1| = \sqrt{2}$ and thus $X_1^0 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$;

2) $\lambda_2 = 3:\ -2x_1 + 2x_2 = 0 \Rightarrow X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $|X_2| = \sqrt{2}$ and thus $X_2^0 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

$$\det\left(X_1^0 \middle| X_2^0\right) = \det\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{1}{2} - \frac{1}{2} = -1 < 0.$$

Therefore,

$$T(\varphi) = \left(X_2^0 \middle| X_1^0\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \cos\varphi = \frac{1}{\sqrt{2}},\ \sin\varphi = \frac{1}{\sqrt{2}},\ \varphi = \frac{\pi}{4}.$$

Consider change of variables corresponding to the turn of exes:

$$x = \frac{x_1 - y_1}{\sqrt{2}},\ y = \frac{x_1 + y_1}{\sqrt{2}}.$$

So

$$x^2 + y^2 + 4xy + 4x + 2y - 5 = 3x_1^2 - y_1^2 + 3\sqrt{2}x_1 - \sqrt{2}y_1 - 5 = 0.$$

After the shift of the origin (It could be found, for example, by completing the squares in x_1 and y_1)

$$\begin{cases} x_2 = x_1 - x_0 = x_1 + 1/\sqrt{2} \\ y_2 = y_1 - y_0 = y_1 + 1/\sqrt{2} \end{cases} \text{ or } \begin{cases} x_1 = x_2 - 1/\sqrt{2} \\ y_1 = y_2 - 1/\sqrt{2} \end{cases}$$

we get

$$3x_1^2 - y_1^2 + 3\sqrt{2}x_1 - \sqrt{2}y_1 - 5 = 3\left(x_1 + 1/\sqrt{2}\right)^2 -$$
$$-\left(y_1 + 1/\sqrt{2}\right)^2 - \frac{3}{2} + \frac{1}{2} - 5 = 3x_2^2 - y_2^2 - 6 = 0$$

or

$$3x_2^2 - y_2^2 = 6.$$

Dividing both sides of this equation by the right-hand member gives

$$\frac{x_2^2}{2} - \frac{y_2^2}{6} = 1.$$

The obtained equation is the canonical equation of the hyperbola with semi-axes $a = \sqrt{2}, b = \sqrt{6}$. Graph of this curve is illustrated on Fig. 77.

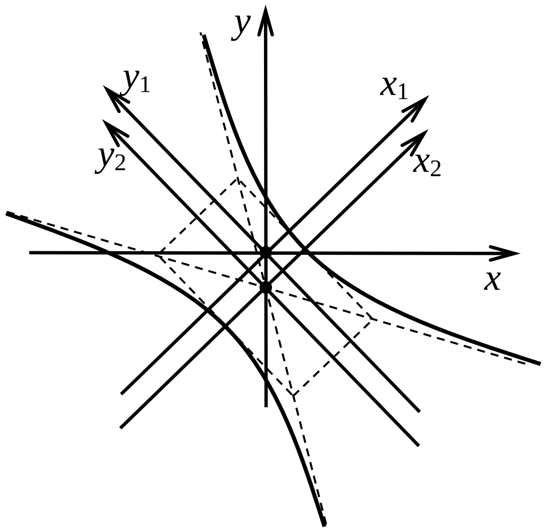


Figure 77

2.4.12. Surfaces of the Second Order

Definition. *A surface of the second order* (or *a quadric*, or *quadric surface*) is a set of points of space that in the Cartesian coordinates is defined as the locus of zeros of a quadratic polynomial, i.e. by the following equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Kx + Ly + Mz + N = 0,$$

where at least one of the coefficients *A*, *B*, *C*, *D*, *E*, *F* is non-zero.

Via translations and rotations every quadric can be transformed to one of several canonical (or normalized) forms. In three-dimensional Euclidean space there are 16 such canonical forms, and the most interesting, the *nondegenerate* forms are given below. The remaining forms are called *degenerate* forms and include planes, lines, points or even no points at all.

I. Cylindrical surfaces (cylinders).

A circular cylinder is one of the most basic curvilinear geometric shapes. That is the surface formed by the points at a fixed distance from a given straight line, *the axis* of the cylinder. The solid enclosed by this surface and by two planes perpendicular to the axis is also called a cylinder.

A cylinder whose cross section is an ellipse, parabola, or hyperbola is called *an elliptic cylinder, parabolic cylinder*, or *hyperbolic cylinder*. More general cylinder is the generalized cylinder, where the cross-section can be any curve.

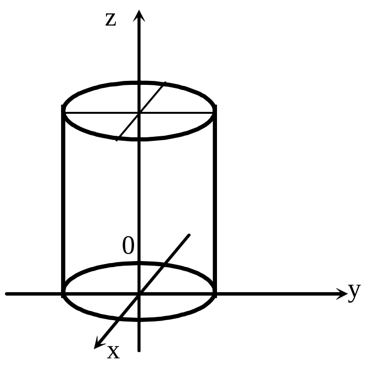


Figure 78

The elliptic cylinder can be defined, for example, by the following equation in Cartesian coordinates (Fig.78):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This equation is for an elliptic cylinder, a generalization of the ordinary, circular cylinder ($a = b$).

The *hyperbolic cylinder* can be defined, for example, by the following equation in Cartesian coordinates (Fig.79):

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

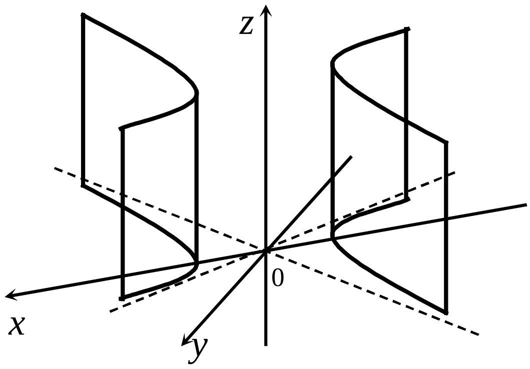


Figure 79

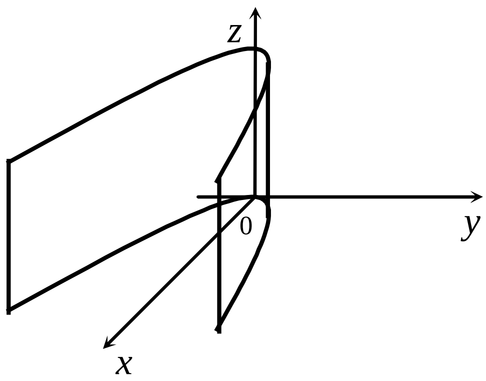


Figure 80

The *parabolic cylinder* can be defined, for example, by the following equation in Cartesian coordinates (Fig.80):

$$x^2=2py \text{ or } y^2=2px.$$

II. Conical surfaces (cones).

A (general) conical surface is the unbounded surface formed by the union of all the straight lines that pass through a fixed point, called the *apex* or the *vertex*, and any point of some fixed space curve, called the *directrix*, that does not contain the apex. Each of those lines is called a *generatrix* of the surface.

The equation of *the elliptic cone* (Fig.81) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}.$$

Circular cone can be obtained from the elliptic cone at $a=b$:

$$x^2+y^2=k^2z^2,$$

where $\kappa^2=a^2/c^2$.

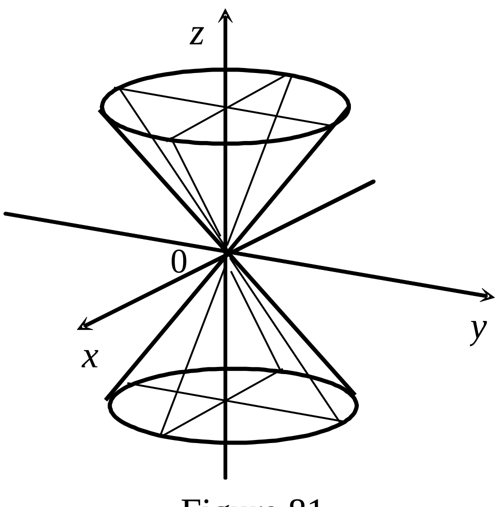


Figure 81

III. Ellipsoids.

An ellipsoid is a type of quadric surface that is a higher dimensional analogue of an ellipse. The equation of a standard ellipsoid body in an *xyz*-Cartesian coordinate system is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

where a and b are the equatorial radii (along the x and y axes) and c is the polar radius (along the z-axis), all of which are fixed positive real numbers determining the shape of the ellipsoid.

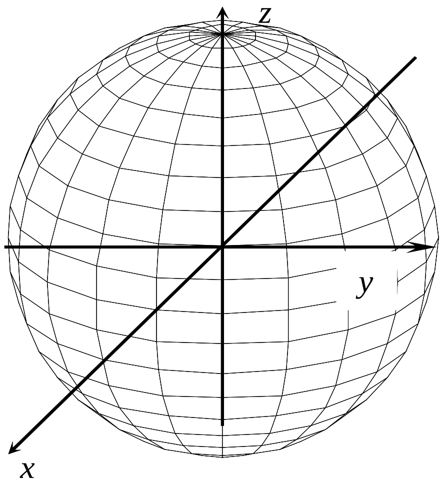


Figure 82

If all three radii are equal, the solid body is a sphere; if two radii are equal, the ellipsoid is a spheroid:

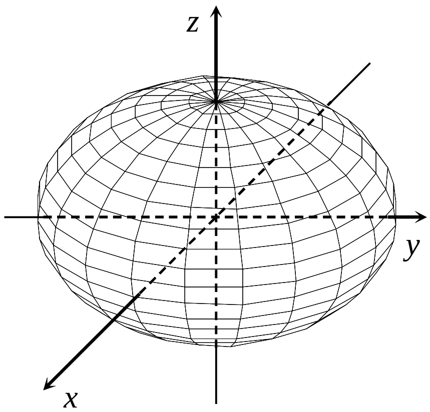


Figure 83

- $a=b=c$ *Sphere* (Fig.82);
- $a=b>c$ *Oblate spheroid* (disk-shaped) (Fig.83);
- $a=b<c$ *Prolate spheroid* (cigar-shaped);
- $a>b>c$ *Scalene ellipsoid* ("three unequal sides").

The points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ lie on the surface and the line segments from the origin to these points are called *the semi-principal axes*. These correspond to the semi-major axis and semi-minor axis of the appropriate ellipses.

IV. Hyperboloids.

A hyperboloid is a quadric, a type of surface in three dimensions, described by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ (hyperboloid of one sheet) (Fig.84)},$$

or

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ (hyperboloid of two sheets) (Fig.85).}$$

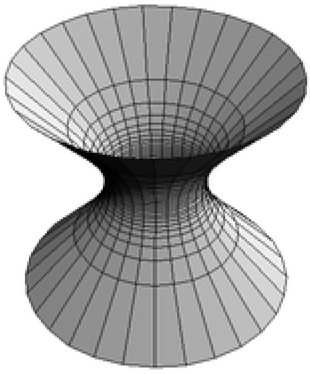


Figure 84

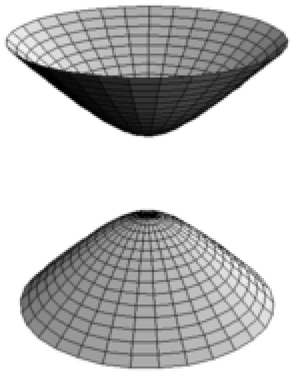


Figure 85

If, and only if, $a = b$, it is *a hyperboloid of revolution*. A hyperboloid of revolution of one sheet can be obtained by revolving a hyperbola around its semi-minor axis. A hyperboloid of revolution of two sheets can be obtained by revolving a hyperbola around its focal axis.

A hyperboloid of one sheet is a doubly ruled surface; if it is a hyperboloid of revolution, it can also be obtained by revolving a line about a skew line.

V. Paraboloids.

A paraboloid is a quadric surface of special kind. There are two kinds of paraboloids, namely *elliptic* and *hyperbolic*. The elliptic paraboloid is shaped like an oval cup and can have a maximum or minimum point. In a suitable coordinate system, it can be represented by the equation

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2} \quad \text{(Fig.86).}$$

This is an elliptical paraboloid which opens upward. In other case it has an equation:

$$z=-\frac{x^2}{a^2}-\frac{y^2}{b^2}.$$

The hyperbolic paraboloid is a doubly ruled surface shaped like a saddle. In a suitable coordinate system, it can be represented by the equation

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad \text{(Fig.87).}$$

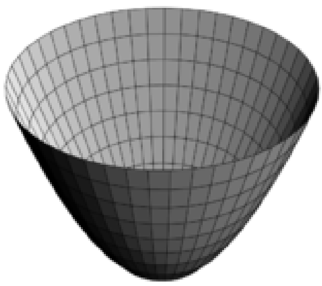


Figure 86

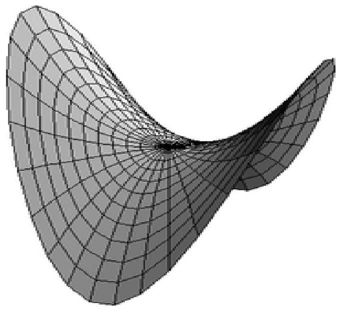


Figure 87

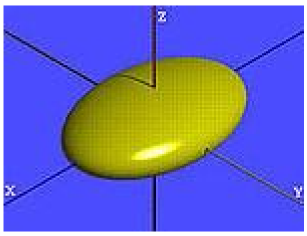
This is a hyperbolic paraboloid that opens up along the *x*-axis and down along the *y*-axis.

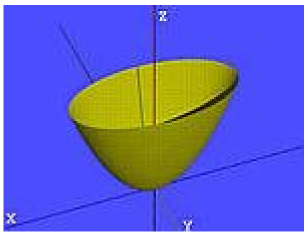
With $a = b$ an elliptic paraboloid is *a paraboloid of revolution*: a surface obtained by revolving a parabola around its axis. It is the shape of the parabolic reflectors used in mirrors, antenna dishes, and the like; and is also the shape of the surface of a rotating liquid, a principle used in liquid mirror telescopes. It is also called *a circular paraboloid*.

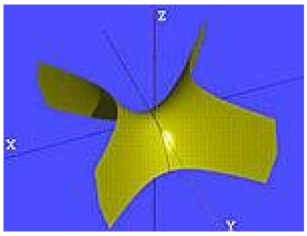
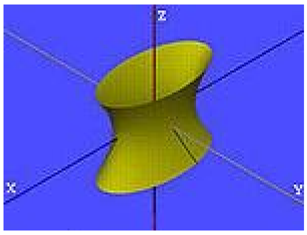
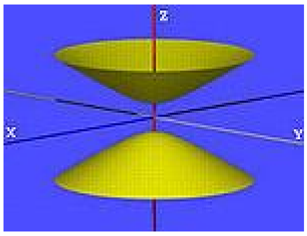
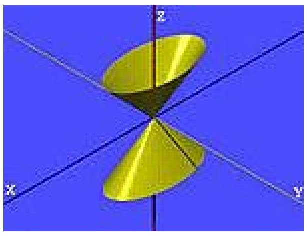
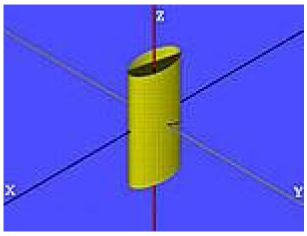
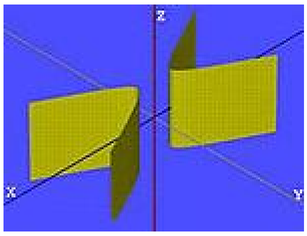
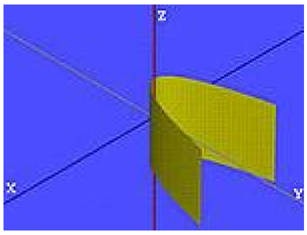
The hyperbolic paraboloid is a ruled surface: it contains two families of mutually skew straight lines. The straight lines in each family are parallel to a common plane, but not to each other.

The information given above is present in the following summary:

ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
spheroid (special case of ellipsoid)	$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$
sphere (special case of spheroid)	$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1$
elliptic paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$
circular paraboloid (special case of elliptic paraboloid)	$\frac{x^2}{a^2} + \frac{y^2}{a^2} - z = 0$





hyperbolic paraboloid	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$	
hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	
hyperboloid of two sheets	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$	
cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	
elliptic cylinder circular cylinder (special case of elliptic cylinder)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$	
hyperbolic cylinder	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	
parabolic cylinder	$x^2=2py$	

2.5. Examples of Problems for Practices on Analytic Geometry

Practice 1 “Concept of Vector. Operations on Vectors”

1. Draw any two uncollinear vectors $\vec{a}$ and $\vec{b}$. Indicate the origin of this vector by point *A*, the terminus of vector by point *B*. Using $\vec{a}$ and $\vec{b}$, draw $2\vec{a}$, $\frac{2}{3}\vec{a}$, $-\vec{a}$, $-3\vec{a}$, $\vec{a}+\vec{b}$, $\vec{a}+2\vec{b}$, $\vec{a}-\vec{b}$, $\vec{b}-\vec{a}$.

2. Consider a parallelogram *ABCD*. Use the vectors $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{AD}$ as basis in plane of the parallelogram. Express all vectors connecting vertex *A* with the middles of the parallelogram sides through the basic vectors.

Answer: Let M_1 be the middle of the side *AB*, M_2 be the middle of the side *BC*, M_3 be the middle of the side *CD*, M_4 be the middle of the side *AD*. Then $\overrightarrow{AM_1}=\frac{1}{2}\vec{a}$, $\overrightarrow{AM_2}=\vec{a}+\frac{1}{2}\vec{b}$, $\overrightarrow{AM_3}=\frac{1}{2}\vec{a}+\vec{b}$, $\overrightarrow{AM_4}=\frac{1}{2}\vec{b}$.

3. Consider a parallelepiped $ABCDA'B'C'D'$. Use vectors $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AD}$ and $\vec{c}=\overrightarrow{AA'}$ as basis in space. Express all vectors connecting vertex *A* with the middles of the parallelepiped sides (or faces) $A'B'C'D'$, $CDD'C'$ and $BCC'B'$ through the basic vectors.

Answer: Let M_1 be the middle of $A'B'C'D'$, M_2 be the middle of $CDD'C'$, M_3 be the middle of $BCC'B'$. Then $\overrightarrow{AM_1}=\frac{1}{2}\left(\vec{a}+\vec{b}\right)+\vec{c}$, $\overrightarrow{AM_2}=\frac{1}{2}\left(\vec{a}+\vec{c}\right)+\vec{b}$, $\overrightarrow{AM_3}=\vec{a}+\frac{1}{2}\left(\vec{b}+\vec{c}\right)$.

4. Consider a hexagon $ABCDEF$. Let. Express the vectors $\overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}, \overrightarrow{FA}$ through $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{BC}$.

Answer: $\overrightarrow{CD}=\vec{b}-\vec{a}$, $\overrightarrow{DE}=-\vec{a}$, $\overrightarrow{EF}=-\vec{b}$, $\overrightarrow{FA}=\vec{a}-\vec{b}$.

5. Consider a parallelogram *ABCD*. Let $\overrightarrow{AK}=\frac{1}{6}\overrightarrow{AC}$, $\overrightarrow{AL}=\frac{1}{5}\overrightarrow{AD}$. Prove that the vectors $\overrightarrow{KB}$ and $\overrightarrow{LK}$ are collinear. *Hint*: use vectors $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{AD}$ as basis.

6. Consider a triangle ABC. Let $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{AC}$. Find any vectors $\vec{d}_1$ and $\vec{d}_2$ directed along the median *AM* and the bisector *AF*.

Answer: $\vec{d}_1=\lambda\left(\vec{a}+\vec{b}\right)$, $\vec{d}_2=\lambda\left(\vec{a}\left|\vec{b}\right|+\vec{b}\left|\vec{a}\right|\right)$, $\lambda\in R$.

7. Let $\left|\vec{a}\right|=6$, $\alpha=120^\circ$, $\beta=45^\circ$, γ be obtuse angle. Find the direction cosines and the Cartesian coordinates of the vector $\vec{a}$.

Answer: $\cos\alpha=-\dfrac{1}{2}$, $\cos\beta=\dfrac{1}{\sqrt{2}}$, $\cos\gamma=-\dfrac{1}{2}$, $\vec{a}=\left(-3,3\sqrt{2},-3\right)$.

8. Let $A(1,2,-1)$, $B\left(4,4,5\right)$. Find the coordinates, the module, direction cosines and the ort of the vector $\vec{a}=\overrightarrow{AB}$.

Answer: $\vec{a}(3,2,6)$, $\left|\vec{a}\right|=7$, $\cos\alpha=\dfrac{3}{7}$, $\cos\beta=\dfrac{2}{7}$, $\cos\gamma=\dfrac{6}{7}$, $\vec{a}^\circ=\left(\dfrac{3}{7},\dfrac{2}{7},\dfrac{6}{7}\right)$.

9. Let $\vec{a}(1,2,-1)$, $\vec{b}\left(4,4,5\right)$. Find the modules of vectors $\vec{a}+\vec{b}$, $2\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$.

Answer: $\left|\vec{a}+\vec{b}\right|=\sqrt{77}$, $\left|2\vec{a}+\vec{b}\right|=\sqrt{109}$, $\left|\vec{a}-\vec{b}\right|=7$.

10. Vectors $\vec{a}=-4\vec{i}+3\vec{j}+\beta\vec{k}$ and $\vec{b}=\vec{i}-\alpha\vec{j}+2\vec{k}$ are collinear. Find α,β.

Answer: $\alpha=\dfrac{3}{4},\beta=-8$.

11. Consider the triangle ABC, where $A(3,2,0)$, $B\left(2,3,-1\right)$, $C\left(-1,3,5\right)$. Find the lengths of median *AM* and of the segments AD_1 and AD_2, where points D_1 and D_2 divide *BC* into 3 equal parts.

Answer: $\dfrac{3}{2}\sqrt{5}$, $\sqrt{19}$, $\sqrt{6}$.

12. Find the decomposition of the vector $\vec{x}(-1,9)$ in the basis of vectors $\vec{g}_1(1,2)$, $\vec{g}_2(-3,5)$.

Answer: $\vec{x}=2\vec{g}_1+\vec{g}_2=(2,1)$.

13. Find the decomposition of the vector $\vec{x}(10,0,6)$ in the basis of vectors $\vec{g}_1(1,-1,0)$, $\vec{g}_2(2,3,1)$, $\vec{g}_3(4,1,5)$.

Answer: $\vec{x}=4\vec{g}_1+\vec{g}_2+\vec{g}_3=(4,1,1)$.

Tasks for self-studying on topic "Concept of Vector. Operations on Vectors"

1	Consider a parallelepiped $ABCDA'B'C'D'$. Use the vectors $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AD}$ and $\vec{c}=\overrightarrow{AA'}$ as basis in space. Express all vectors connecting vertex A with the middles of the parallelepiped edges.
2	Let $\lvert\vec{a}\rvert=2$, $\alpha=60^\circ$, $\gamma=45^\circ$, β be acute angle. Find the direction cosines and the Cartesian coordinates of the vector $\vec{a}$.
3	Consider the parallelogram *ABCD*, where $A(1,5,2)$, $B(2,3,-1)$, $C(0,4,1)$. Find the coordinates of the vertex *D*.
4	Let $\vec{a}=2\vec{i}+5\vec{j}+\beta\vec{k}$ and $\vec{b}=2\vec{i}+\alpha\vec{j}+3\vec{k}$. Find α,β if $\vec{a}+\vec{b}$ and $\vec{b}$ are collinear.
5	Find the decomposition of the vector $\vec{x}(1,-8,9)$ in the basis of vectors $\vec{g}_1(1,1,2)$, $\vec{g}_2(1,3,0)$, $\vec{g}_3(2,-1,5)$.

Answers: **1.** $\frac{1}{2}\vec{a},\frac{1}{2}\vec{b},\frac{1}{2}\vec{c}$, $\frac{1}{2}\vec{c}+\vec{a},\frac{1}{2}\vec{b}+\vec{a}$, $\frac{1}{2}\vec{c}+\vec{b},\frac{1}{2}\vec{a}+\vec{b}$, $\frac{1}{2}\vec{a}+\vec{c},\frac{1}{2}\vec{b}+\vec{c}$, $\frac{1}{2}\vec{a}+\vec{b}+\vec{c}$, $\vec{a}+\frac{1}{2}\vec{b}+\vec{c}$, $\vec{a}+\vec{b}+\frac{1}{2}\vec{c}$; **2.** $\vec{a}(1,1,\sqrt{2})$; **3.** $D(-1,6,4)$; **4.** $\alpha=\frac{5}{3}$, $\beta=3$; **5.** $\vec{x}=2\vec{g}_1-3\vec{g}_2+\vec{g}_3=(2,-3,1)$.

Practice 2 "Scalar Product"

1. Let $\lvert\vec{a}\rvert=2$, $\lvert\vec{b}\rvert=3$, the angle α between $\vec{a}$ and $\vec{b}$ be equal to 60°. Find $(\vec{a},\vec{b})$, $(\vec{a},2\vec{b})$, $(\vec{a}+\vec{b},\vec{b})$, $(\vec{a}+\vec{b},\vec{a}+\vec{b})$, $(2\vec{a}+3\vec{b},\vec{a}-\vec{b})$.

Answer: 3, 6, 12, 19, -16.

2. Let $\lvert\vec{a}\rvert=1$, $\lvert\vec{b}\rvert=4$, the angle α between $\vec{a}$ and $\vec{b}$ be equal to 45°. Find the length of the vector $3\vec{a}-\vec{b}$.

Answer: $\sqrt{25-12\sqrt{2}}$.

3. Find the lengths of diagonals of the parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$ if $\vec{a}=5\vec{p}+2\vec{q},\ \vec{b}=\vec{p}-3\vec{q};\ |\vec{p}|=2\sqrt{2},\ |\vec{q}|=3,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

Answer: $15, \sqrt{593}$.

4. Let $|\vec{a}|=2$, $|\vec{b}|=3$, the angle α between $\vec{a}$ and $\vec{b}$ be equal to $\frac{\pi}{3}$. Find the value of β, if $\vec{a}+\beta\vec{b}\perp\vec{a}-\vec{b}$.

Answer: $\beta=\frac{1}{6}$.

5. Find the angle between two unit vectors $\vec{a}$ and $\vec{b}$ if vectors $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular.

Answer: 60°.

6. Find $pr_{\vec{c}}\left(\vec{a}+\vec{b}\right)$, if $\vec{a}=3\vec{i}-6\vec{j}-\vec{k}$, $\vec{b}=\vec{i}+4\vec{j}-5\vec{k}$, $\vec{c}=3\vec{i}-4\vec{j}+12\vec{k}$.

Answer: -4.

7. Consider the triangle ABC, where $A(2,1)$, $B(4,-5)$, $C(5,2)$. Find the angles of this triangle.

Answer: $\angle A=\frac{\pi}{2}, \angle B=\arccos\frac{2\sqrt{5}}{5}, \angle C=\arccos\frac{\sqrt{5}}{5}$.

8. Find the vector $\vec{b}$ collinear to vector $\vec{a}(1,3,-5)$, if $(\vec{a},\vec{b})=-70$.

Answer: $\vec{b}(-2,-6,10)$.

9. Let $\vec{a}(1,1,-3)$, $\vec{b}(0,1,-1)$, $\vec{c}(2,1,5)$. Find the vector $\vec{x}$, if $\vec{x}\perp\vec{a}$, $\vec{x}\perp\vec{b}$, $(\vec{x},\vec{c})=10$.

Answer: $\vec{x}(2,1,1)$.

10. Consider a triangle ABC with $\vec{a}=\overrightarrow{AB}(5,2)$, $\vec{b}=\overrightarrow{AC}(2,-4)$ and the altitude *BD*. Find the vector $\overrightarrow{BD}$ and its module.

Answer: $\overrightarrow{BD}(-4.8,-2.4)$, $\left|\overrightarrow{BD}\right|=\frac{12\sqrt{5}}{5}$.

11. Consider a rhomb $ABCD$ with $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AD}$. Prove that the diagonals of the rhomb are always perpendicular. *Hint:* express the diagonals through $\vec{a}$ and $\vec{b}$.

Tasks for self-studying on topic "Scalar Product"

1	Find cosine of angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ if $A(1,-2,3),\ B(-1,0,5),\ C(-1,1,4)$.
2	Find projection of the vector $\vec{a}$ on the direction of the vector $\vec{b}$ if $\vec{a}=\vec{p}-2\vec{q},\ \vec{b}=2\vec{p}+\vec{q};\ \ \lvert\vec{p}\rvert=2,\ \ \lvert\vec{q}\rvert=3,\ \ (\vec{p}\,\hat{}\,\vec{q})=3\pi/4$.
3	Find decomposition of the vector $\vec{c}$ in the basis of vectors $\vec{a}$ and $\vec{b}$ if $\lvert\vec{a}\rvert=3,\ \lvert\vec{b}\rvert=2,\ (\vec{a}\,\hat{}\,\vec{b})=\dfrac{\pi}{3},\ (\vec{a},\vec{c})=3,\ (\vec{b},\vec{c})=-5$.
4	Find α, if $\vec{a}=\alpha\vec{i}+5\vec{j}-2\vec{k}$ and $\vec{b}=3\vec{i}+\vec{j}-\alpha\vec{k}$ are perpendicular.
5	Find vector $\vec{x}$ if $(\vec{x},\vec{a})=-5,\ (\vec{x},\vec{b})=-11,\ (\vec{x},\vec{c})=20,\ \vec{a}=2\vec{i}-\vec{j}+3\vec{k},\ \vec{b}=\vec{i}-3\vec{j}+2\vec{k},\ \vec{c}=3\vec{i}+2\vec{j}-4\vec{k}$.

Answers: **1.** $\arccos\dfrac{\sqrt{42}}{7}$; **2.** $\dfrac{9\sqrt{2}-10}{\sqrt{25-12\sqrt{2}}}$; **3.** $\vec{c}=\vec{a}-2\vec{b}$; **4.** $\alpha=-1$; **5.** $\vec{x}(2,3,-2)$.

Practice 3 "Vector and Mixed Products"

1. Let $\lvert\vec{a}\rvert=2,\ \lvert\vec{b}\rvert=3,\ (\vec{a},\vec{b})=4$. Find $\lvert\vec{a}\times\vec{b}\rvert$.

Answer: $2\sqrt{5}$.

2. Find the area of the parallelogram constructed on vectors $\vec{a}$ and $\vec{b}$ if $\vec{a}=\vec{p}+2\vec{q},\ \vec{b}=3\vec{p}-\vec{q};\ \ \lvert\vec{p}\rvert=\sqrt{2},\ \ \lvert\vec{q}\rvert=3,\ \ (\vec{p}\,\hat{}\,\vec{q})=\pi/4$.

Answer: 21.

3. Find the altitude CD of the triangle ABC if $\overrightarrow{AB}=3\vec{p}-4\vec{q},\ \ \overrightarrow{BC}=\vec{p}+5\vec{q}$, where $\vec{p}$ and $\vec{q}$ are perpendicular unit vectors.

Answer: 3.8.

4. Find the value of α if $\alpha\vec{a}+5\vec{b}\parallel 3\vec{a}-\vec{b}$, where $\vec{a}$ and $\vec{b}$ are not collinear.

Answer: $\alpha=-15$.

5. Prove that $\left[\vec{a},\vec{b}\times\vec{c}\right]=0$ if $\vec{a}\perp\vec{b}$, $\vec{a}\perp\vec{c}$.

6. Let $\vec{a}=\vec{i}+2\vec{j}+3\vec{k}$, $\vec{b}=-\vec{i}+4\vec{j}+2\vec{k}$, $\vec{c}=3\vec{i}+2\vec{j}+\vec{k}$, $\vec{d}=\vec{i}+4\vec{k}$. Find $\vec{a}\times\vec{b}$, $\vec{a}\times\vec{c}$, $\vec{b}\times\vec{d}$, $3\vec{a}\times\left(\vec{c}+\vec{d}\right)$.

Answer: $\vec{a}\times\vec{b}=(-8,-5,6)$, $\vec{a}\times\vec{c}=(-4,8,-4)$, $\vec{b}\times\vec{d}=(16,6,-4)$, $3\vec{a}\times\left(\vec{c}+\vec{d}\right)=(12,21,-18)$.

7. Find the area and the altitude of the triangle *ABC* dropped from the vertex *B* if $A(1,2,3)$, $B(-1,4,2)$, $C(0,3,4)$.

Answer: $S=\dfrac{3\sqrt{2}}{2}$, $h=\sqrt{6}$.

8. Find the area of the triangle ABC, where $A(1,2)$, $B(-1,4)$, $C(2,3)$.

Answer: $S=2$.

9. Find the vector $\vec{x}$ perpendicular to vectors $\vec{a}(1,2,1)$ and $\vec{b}(3,1,3)$ if $|\vec{x}|=10$ and the angle between $\vec{x}$ and positive semi-axis Oz is acute.

Answer: $\vec{x}\left(-5\sqrt{2},0,5\sqrt{2}\right)$.

10. Find the volume of the parallelepiped constructed on the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ if:

1) $\vec{a}=\vec{p}-3\vec{q}+\vec{r}$, $\vec{b}=2\vec{p}+\vec{q}-3\vec{r}$, $\vec{c}=\vec{p}+2\vec{q}+\vec{r}$, where $\left(\vec{p},\vec{q},\vec{r}\right)=1$;

2) $\vec{a}=4\vec{p}-5\vec{q}$, $\vec{b}=2\vec{p}-\vec{q}$, $\vec{c}=\vec{p}+2\vec{q}$, where $\left(\vec{p},\vec{q},\vec{r}\right)=1$;

3) $\vec{a}=\vec{i}-3\vec{j}+2\vec{k}$, $\vec{b}=4\vec{i}+\vec{j}+3\vec{k}$, $\vec{c}=\vec{i}+\vec{j}+\vec{k}$.

Answer: 1) $V=25$; 2) $V=0$; 3) $V=7$.

11. Find the altitude of the tetrahedron *ABCD* dropped on the base *ABC* if $A(1,2,-1), B(0,1,3), C(2,1,1), D(0,0,4)$.

Answer: $h=\dfrac{2\sqrt{11}}{11}$.

12. Check if these vectors are complanar:

1) $\vec{a}=\vec{i}-2\vec{j}+\vec{k}$, $\vec{b}=3\vec{i}+\vec{j}-2\vec{k}$, $\vec{c}=7\vec{i}+14\vec{j}-13\vec{k}$;

2) $\vec{a}=4\vec{i}+\vec{j}-3\vec{k}$, $\vec{b}=\vec{i}-4\vec{j}+\vec{k}$, $\vec{c}=3\vec{i}-2\vec{j}+2\vec{k}$;

3) $\vec{a}=\vec{p}\times\vec{m}$, $\vec{b}=\vec{q}\times\vec{m}$, $\vec{c}=\vec{r}\times\vec{m}$, where $\vec{p},\ \vec{q},\ \vec{r}$ are mutually perpendicular.

Answer: 1) complanar; 2) not complanar; 3) complanar vectors.

13. Prove that the given four points belong to the same plane: $A(-2,0,2), B(1,-3,-1), C(2,1,8), D(5,-1,7)$.

Tasks for self-studying on topic "Vector and Mixed Products"

1	Find an area of the triangle constructed on vectors $\vec{a}(1,1,3)$ and $\vec{b}(-1,2,0)$ and the length of altitude dropped on the side of the vector $\vec{a}$.
2	Find an area of the parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$ if $\vec{a}=\vec{p}-2\vec{q},\ \vec{b}=2\vec{p}+\vec{q};\ \lvert\vec{p}\rvert=2,\ \lvert\vec{q}\rvert=3,\ \left(\vec{p}\,\hat{}\,\vec{q}\right)=3\pi/4.$
3	Find the volume of the tetrahedron $ABCD$ if $A(2,3,-1)$, $B(2,-2,4)$, $C(-1,1,3)$, $D(1,1,2)$.
4	Check that these points belong to the same plane: $A(1,0,7)$, $B(-1,-1,2)$, $C(2,-2,2)$, $D(0,1,9)$.
5	Find the unit vector $\vec{x}$ perpendicular to the vector $\vec{a}=(4;-2;-3)$ and the axis *Oy* if it forms an obtuse angle with the axis *Ox*.

Answers: **1.** $S=\dfrac{3\sqrt{6}}{2}$, $h=\dfrac{3\sqrt{66}}{11}$; **2.** $15\sqrt{2}$; **3.** $V=\dfrac{5}{6}$; **4.** True; **5.** $\vec{x}\left(-\dfrac{3}{5},0,-\dfrac{4}{5}\right)$.

Practice 4 "Plane in Space"

1. Points $M_1(2,-1,2)$ and $M_2(3,2,-1)$ are given. Find an equation of the plane passing through the point M_1 and perpendicular to the vector $\overrightarrow{M_1M_2}$.

Answer: $x+3y-3z+7=0$.

2. Find an equation of the plane:

1) passing through the point $M_1(2,-3,5)$ and parallel to plane Oxy;

2) passing through the point $M_2(1,2,4)$ and parallel to plane Oxz;

3) passing through the point $M_3(1,2,7)$ and parallel to plane Oyz;

Answer: 1) $z-5=0$; 2) $y-2=0$; 3) $x-1=0$.

3. Find an equation of the plane passing through the points $M_1(2,-1,3)$, $M_2(3,1,2)$ and parallel to the vector $\vec{a}(3,-1,4)$.

Answer: $x-y-z=0$.

4. Find an equation of the plane passing through the origin and perpendicular to the planes $x-4y+3z-1=0$ and $3x+2y-z=0$.

Answer: $x-5y-7z=0$.

5. Find an equation of the plane passing through the points $M_1(3,-1,2)$, $M_2(0,1,-1)$ and $M_3(1,0,2)$.

Answer: $3x+6y+z-5=0$.

6. Determine which pair of equations represents perpendicular planes:

1) $3x-y-2z+7=0$, $x+9y-3z-4=0$;

2) $2x+3y-z+6=0$, $x-y-z+5=0$;

3) $2x-5y+z-4=0$, $x+2z-3+1=0$.

Answer: 1) and 2).

7. What values of parameters l and m correspond to the equations of parallel planes: $2x+ly+3z+1=0$, $mx-6y-6z-5=0$?

Answer: $l=3, m=-4$.

8. Determine the angles obtained at intersection of the planes $x-y\sqrt{2}+z-1=0$, $x+y\sqrt{2}-z+3=0$.

Answer: $\pi/3$ and $2\pi/3$.

9. It is known that some plane passes through the point $M_0(6, 5, -4)$ and cuts an intercept on axis Ox equal to $a=-3$ and an intercept on axis Oz equal to $c=2$. Find an equation of this plane.

Answer: $\dfrac{x}{-3}+\dfrac{y}{1}+\dfrac{z}{2}=1$.

10. Find the distance between two parallel planes: $x-2y-2z-15=0$, $x-2y-2z-6=0$.

Answer: $d=3$.

11. Find the equations of two planes parallel to the plane $3x-6y-2z+14=0$ and distant from it on 3 units.

Answer: $3x-6y-2z+35=0, 3x-6y-2z-7=0$.

Tasks for self-studying on topic “Plane in Space”

1	Find an equation of the plane passing through the point $M(1,-1,2)$ and parallel to the vectors $\vec{a}_1=(3,-1,4)$, $\vec{a}_2=(5,0,-1)$.
2	Find an equation of the plane passing through two points $M_1(1,-1,2)$ and $M_2(2,1,3)$ and perpendicular to the plane $x+y+3z-5=0$.
3	Find the values of parameters l and m corresponding to the equations of parallel planes: $4x-y-lz-9=0$, $2x+my+2z-3=0$.
4	Find distance between two parallel planes: $2x-3y+6z-21=0$, $2x-3y+6z+14=0$.
5	Find an equation of the plane that passes through the point $M_0(2,3,-4)$ and cuts equal nonzero intercepts from the axes of coordinates.

Answers: **1.** $x+23y+5z+12=0$; **2.** $5x-2y-z-5=0$; **3.** $l=-4,\ m=-1/2$; **4.** $d=5$; **5.** $x+y+z=1$.

Practice 5 “Straight Line and Plane in Space”

1. Find the canonical equations of the straight line passing through the point $M_0(2,0,-3)$ and parallel to:

1) the vector $\vec{a}(2,1,5)$;

2) the straight line $\dfrac{x-1}{4}=\dfrac{y+2}{2}=\dfrac{z+1}{-1}$;

3) the axis Ox;

4) the axis Oy;

5) the axis Oz.

Answer: 1) $\dfrac{x-2}{2}=\dfrac{y}{1}=\dfrac{z+3}{5}$; 2) $\dfrac{x-2}{4}=\dfrac{y}{2}=\dfrac{z+3}{-1}$; 3) $\dfrac{x-2}{1}=\dfrac{y}{0}=\dfrac{z+3}{0}$; 4) $\dfrac{x-2}{0}=\dfrac{y}{1}=\dfrac{z+3}{0}$; 5) $\dfrac{x-2}{0}=\dfrac{y}{0}=\dfrac{z+3}{1}$.

2. Points $A(3,6,-7)$, $B(-5,2,3)$, $C(1,-2,3)$ are the vertices of the triangle. Find the canonical equations of the triangle sides and the parametric equations of the triangle median dropped from the vertex C.

Answer: $\frac{x-3}{-8}=\frac{y-6}{-4}=\frac{z+7}{10}$, $\frac{x-3}{2}=\frac{y-6}{8}=\frac{z+7}{-10}$, $\frac{x-1}{6}=\frac{y+2}{-4}=\frac{z-3}{0}$, $x=2t+1, y=-6t-2,\ z=5t+3$.

3. Find the canonical equations of the straight line $\begin{cases} x-2y+3z-4=0 \\ 3x+2y-5z+2=0 \end{cases}$.

Answer: $\frac{x-1}{2}=\frac{y}{7}=\frac{z-1}{4}$.

4. Find the canonical equations of the straight line passing through the point $M_0(2,0,3)$, parallel to the plane $x+y+z+3=0$ and perpendicular to the axis Oz.

Answer: $\frac{x-2}{1}=\frac{y}{-1}=\frac{z-3}{0}$.

5. Prove that these straight lines are parallel: $\frac{x+2}{3}=\frac{y-1}{-2}=\frac{z}{1}$ and $\begin{cases} x+y-z=0 \\ x-y-5z-8=0 \end{cases}$.

6. Prove that these straight lines intersect each other: $x=4t-3$, $y=7t-2$, $z=-4t+3$ and $x=t-1, y=-4t+13,\ z=-t+1$.

Hint: this task could be solved, for example, in two ways: by means of the condition of straight line intersection and by means of finding the point of intersection.

7. Find the equations of the straight line passing through the point $M_1(-1,2,-3)$, perpendicular to the vector $\vec{a}(6,-2,-3)$ and intersecting the straight line $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-3}{-5}$.

Answer: $\frac{x+1}{2}=\frac{y-2}{-3}=\frac{z+3}{6}$.

8. Find the point of intersection of the straight line and plane: $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z}{6}$, $2x+3y+z-7=0$.

Answer: $(5,-5,12)$.

9. Find an equation of the plane passing through the point $M_0(1,-2,1)$ and perpendicular to straight line $\begin{cases} x-2y+z+4=0 \\ x+y-z+5=0 \end{cases}$.

Answer: $x+2y+3z=0$.

10. Find a projection of the point $P(2,-1,3)$ on the straight line $x=3t, y=5t-7, z=2t+2$.

Answer: $(3,-2,4)$.

11. Find the point Q symmetrical to the point $P(1,3,-4)$ relatively the plane $3x+y-2z=0$.

Answer: $Q(-5,1,0)$.

12. It is known that the straight line $x=4+3t, y=2-4t, z=t$ lies in the plane $Ax+2y-4z+D=0$. Find values A and D.

Answer: $A=4, D=-20$.

13. Find the distance d from point $P(3,1,-2)$ to the straight line $\frac{x+1}{2}=\frac{y+2}{1}=\frac{z}{2}$.

Answer: $d=\frac{2\sqrt{53}}{3}$.

14. Check that these straight lines are parallel and find the distance d between them: $\begin{cases} 2x+2y-z-10=0 \\ x-y-z-22=0 \end{cases}$; $\frac{x+7}{3}=\frac{y-5}{-1}=\frac{z-9}{4}$.

Answer: $d=25$.

15. Find the shortest distance between two straight lines: $\frac{x+7}{3}=\frac{y+4}{4}=\frac{z+3}{-2}$, $\frac{x-21}{6}=\frac{y+5}{-4}=\frac{z-2}{-1}$.

Answer: 13.

16. Find the parametric equations of the common perpendicular of two straight lines: $x=3t-7, y=-2t+4, z=3t+4$ and $x=t+1, y=2t-8, z=-t-12$.

Answer: $x=2t-5, y=-3t+1, z=-4t$.

Tasks for self-studying on topic "Straight Line and Plane in Space"

1	Find the canonical equations of the given straight line: $\begin{cases} 2x+y+4z-4=0; \\ 2x-y-3z=0. \end{cases}$
2	Let $A(3,-1,-1)$, $B(1,2,-7)$, $C(-5,14,-3)$ be the vertices of some triangle. Find the canonical equations of the median AD.
3	Find the canonical equations of the straight line passing through the origin and parallel to two planes $2x-y+z=0$ and $x+3y+2z-5=0$.
4	Find an equation of plane passing through the point $A(2,-1,1)$ if the straight line $\frac{x}{1}=\frac{y-2}{-3}=\frac{z}{1}$ lies in that plane.
5	Find value C if straight line $\begin{cases} 3x-2y+z+2=0 \\ 2x-5y+4z+7=0 \end{cases}$ is parallel to plane $-4x-y+Cz-2=0$.
6	Find a point of intersection of the given straight line and plane: $\frac{x-3}{1}=\frac{y-1}{-1}=\frac{z+5}{2}$, $x+7y+3z+5=0$.
7	Find a distance d from the point $P(2,3,-1)$ to the straight line $\frac{x-5}{3}=\frac{y}{2}=\frac{z+25}{-2}$.

Answers: 1. $\frac{x-1}{1}=\frac{y-2}{14}=\frac{z}{-4}$; **2.** $\frac{x-3}{5}=\frac{y+1}{-9}=\frac{z+1}{4}$; **3.** $\frac{x}{-5}=\frac{y}{-3}=\frac{z}{7}$;
4. $y+3z-2=0$; **5.** $C=2$; **6.** $(4,0,-3)$; **7.** 21.

Practice 6 "Straight Line in Plane"

1. Find an equation of the straight line passing through the points $A(3,4)$ and $B(2,-2)$. Plot the graph of this line and find the points of intersection of this straight line with the coordinate axes.

Answer: $y = 6x - 14$; $(0,-14)$, $(7/3, 0)$.

2. Find an equation of the straight line passing through the point $P(1,1)$ and 1) parallel to the straight line $4x + 3y - 1 = 0$; 2) perpendicular to the straight line $4x + 3y - 1 = 0$.

Answer: 1) $4x + 3y - 7 = 0$; 2) $3x - 4y + 1 = 0$.

3. Coordinates of the triangle vertices are given: $A(1,1)$, $B(2,-1)$, $C(3,-2)$.
Find:

a) the equations of the triangle sides;
b) the equation of the triangle median dropped from the vertex B;
c) the equation of the altitude dropped from the vertex A;
d) the equation of the bisector of the angle at vertex C;
e) the equation of the middle line of the triangle parallel to the side CB;
f) the coordinates of the center of gravity of this triangle;
g) a value of the altitude dropped from the vertex B;
h) an area of the triangle ABC.

Hint: the equation of a side (median, altitude, etc.) means the equation of a straight line passing through this side.

Answers:

a) AB: $2x + y - 3 = 0$; AC: $3x + 2y - 5 = 0$; CB: $x + y - 1 = 0$;
b) $x = 2$;
c) $y = x$;
d) $(3\sqrt{2} + \sqrt{13})x + (2\sqrt{2} + \sqrt{13})y - 5\sqrt{2} - \sqrt{13} = 0$;
e) $2x + 2y - 3 = 0$;
f) $(2; -2/3)$;
g) $1/\sqrt{13}$;
h) $1/2$.

4. Find the point M symmetrical to the point $P(1,-4)$ relatively to the straight line passing through the points $A(-4,1)$ and $B(4,-3)$.

Answer: $M(3,0)$. Straight line is $x+2y+2=0$.

5. Find an equation of the straight line that passes through the point $P(2;3)$ and cuts intercepts from the coordinate axes of the same length.

Answer: $x+y-5=0,\ x-y+1=0$.

6. Find an angle between two straight lines: $5x-y+3=0$, $3x+2y-5=0$.

Answer: $\pi/4$.

7. Find an equation of the straight line parallel to the following two straight lines and equidistant from them: $x-2y+5=0$, $x-2y-15=0$.

Answer: $x-2y-5=0$.

8. Find the vertices of the rectangular $ABCD$, if the equations of two its sides $x-2y=0;\ x-2y+15=0$ and the equation of its diagonal $7x+y-15=0$ are given.

Answer: $A(2,1),\ B(4,2),\ C(-1,7),\ D(1,8)$.

9. Two vertices of the triangle $A(2,1)$ and $B(5,3)$ and the point of intersection of its altitudes $H(1,0)$ are given. Find equations of the triangle sides.

Answer: $2x-y-3=0,\ x+y-8=0,\ 4x+3y-11=0$.

10. Find the equation of straight line passing through the point $M_0(4,7)$ and creating the angle 45° with the straight line $2x+3y+1=0$.

Answer: $x-5y+31=0,\ 5x+y-27=0$.

11. Find the equations of the triangle sides if the vertex $A(4,-1)$ and the equations of two its bisectors $x-1=0$ and $x-y-1=0$ are given.

Answer: $2x-y+3=0,\ 2x+y-7=0,\ x-2y-6=0$.

12. Find the equations of the triangle sides if the vertex $B(2,-1)$, the equation of the altitude $3x-4y+27=0$ and the equation of the bisector $x+2y-5=0$ dropped from different vertices are given.

Answer: $4x+7y-1=0,\ y-3=0,\ 4x+3y-5=0$.

Tasks for self-studying on topic “Straight Line in Plane”

1	Coordinates of the triangle vertices are given: $A(0,1)$, $B(-2,2)$, $C(2,1)$. Find: a) the equations of the triangle sides; b) the equation of the triangle median dropped from the vertex B; c) the equation of the altitude dropped from the vertex A; d) the equation of the bisector of the angle at vertex C; e) the equation of the middle line of the triangle parallel to the side CB; f) the coordinates of the center of gravity of this triangle; g) a value of the altitude dropped from the vertex B; h) an area of the triangle ABC.
2	Find an equation of the straight line passing through the point $P(-4,6)$ and cutting in the first coordinate quarter the triangle with area equal to 6.
3	Find a point on the axis Oy equidistant from the origin and the straight line $3x-4y+12=0$.

Answers:

1.

a) AB: $x+2y-2=0$; AC: $y=1$; CB: $x+4y-6=0$;

b) $x+3y-4=0$;

c) $4x-y+1=0$;

d) $x+(4+\sqrt{17})y-\sqrt{17}-6=0$;

e) $x+4y-5=0$;

f) $(0,4/3)$;

g) 1;

h) 1;

2. $3x+4y=12$; **3.** $M_1(0,-12)$, $M_2(0,4/3)$.

Practice 7 "The Second Order Curves"

1. Plot graphs of the following curves of the second order. Determine their type and main characteristics:

1) $x^2 + y^2 = 25$; 2) $x^2 + y^2 = 0$; 3) $(x-2)^2 + (y-1)^2 = 9$;

4) $x^2 = 4y$; 5) $x^2 = -9y$; 6) $y^2 = -4x$;

7) $(x-2)^2 = 2(y+1)$; 8) $\frac{x^2}{9} + \frac{y^2}{4} = 1$; 9) $\frac{x^2}{4} + \frac{y^2}{25} = 1$;

10) $\frac{x^2}{1} - \frac{y^2}{4} = 1$; 11) $\frac{y^2}{9} - \frac{x^2}{4} = 1$; 12) $-\frac{(x-1)^2}{4} + (y-2)^2 = 1$.

Answer: 1) circle, $R = 5$; 2) point $(0,0)$; 3) circle with center (2,1) and $R = 3$; 4) parabola with axis of symmetry parallel to O*y* and $p = 2$; 5) parabola with axis of symmetry parallel to O*y* and $p = -9/2$; 6) parabola with axis of symmetry parallel to O*x* and $p = -2$; 7) parabola with axis of symmetry parallel to O*y*, center $(2,-1)$ and $p = 1$; 8) ellipse, $a = 3, b = 2$; 9) ellipse, $a = 2, b = 5$; 10) hyperbola, $a = 1, b = 2$; 11) conjugate hyperbola, $a = 2, b = 3$; 12) conjugate hyperbola with center $(1,2)$, $a = 2, b = 1$.

2. Reduce the equations of the second order curves to the canonical form and plot graphs of these curves. *Hint*: tasks are in the second column of table, answers are in the third.

#	*The second order curve*	*Its type and canonical equation*
1)	$x^2 + y^2 - 6x - 2y + 6 = 0$;	circle $(x-3)^2 + (y-1)^2 = 4$
2)	$x^2 + y^2 + 4x - 2y + 4 = 0$;	circle $(x+2)^2 + (y-1)^2 = 1$
3)	$x^2 + y^2 = 8x$;	circle $(x-4)^2 + y^2 = 16$
4)	$x^2 + 6x - 4y + 17 = 0$;	parabola $(x+3)^2 = 4(y-2)$
5)	$x^2 - 6x + 10y - 1 = 0$;	parabola $(x-3)^2 = -10(y-1)$

6)	$y^2+2x-6y-1=0$;	parabola $(y-3)^2=-2(x-5)$
7)	$y^2-2x-2y-3=0$;	parabola $(y-1)^2=2(x+2)$;
8)	$9x^2+4y^2-18x-16y-11=0$;	ellipse $\frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1$;
9)	$x^2+25y^2-10x+50y+25=0$;	ellipse $\frac{(x-5)^2}{25}+\frac{(y+1)^2}{1}=1$;
10)	$9x^2+16y^2-18x+64y-71=0$;	ellipse $\frac{(x-1)^2}{16}+\frac{(y+2)^2}{9}=1$;
11)	$x^2-y^2-4x-2y+2=0$;	hyperbola $(x-2)^2-(y+1)^2=1$;
12)	$x^2-4y^2-2x+16y-19=0$;	hyperbola $\frac{(x-1)^2}{4}-(y-2)^2=1$;
13)	$-4x^2+9y^2+40x+18y-127=0$;	c. hyperbola $-\frac{(x-5)^2}{9}+\frac{(y+1)^2}{4}=1$;
14)	$-16x^2+y^2-160x+8y-400=0$;	c. hyperbola $-(x+5)^2+\frac{(y+4)^2}{16}=1$;
15)	$-9x^2+4y^2-16y-20=0$;	c. hyperbola $-\frac{x^2}{4}+\frac{(y-2)^2}{9}=1$;
16)	$-4x^2+y^2+16x-2y-15=0$;	two intersecting straight lines $y-1=\pm 2(x-2)$;
17)	$y^2-4y+3=0$;	two parallel straight lines $y=1,\ y=3$;
18)	$7x^2+y^2-14x-6y+16=0$;	point $(1,3)$.

3. Reduce the equations of the second order curves to the canonical form and plot graphs of these curves:

1) $11x^2-10xy\sqrt{3}+y^2+16=0$;

2) $13x^2 - 10xy + 13y^2 - 72 = 0$;

3) $3x^2 + 2\sqrt{3}xy + y^2 + 8x - 8\sqrt{3}y = 0$;

4) $x^2 - 2xy + y^2 - 8\sqrt{2}x = 0$.

Answer: 1) hyperbola $\frac{x'^2}{4} - y'^2 = 1$, $\varphi = \frac{\pi}{3}$; 2) ellipse $\frac{x'^2}{9} + \frac{y'^2}{4} = 1$, $\varphi = \frac{\pi}{4}$; 3) parabola $x'^2 = 4y'$, $\varphi = \frac{\pi}{6}$; 4) parabola $(y'+2)^2 = 4(x'+1)$, $\varphi = \frac{\pi}{4}$.

Tasks for self-studying on topic "The Second Order Curves"

1	Plot graphs of the following curves of the second order. Determine their type and main characteristics: 1) $(x+4)^2 + (y-3)^2 = 5$; 2) $\frac{(x+2)^2}{4} + \frac{(y-1)^2}{16} = 1$; 3) $(x-1)^2 = 6(y+3)$; 4) $(y-2)^2 = -8(x+1)$; 5) $\frac{(x-1)^2}{9} - \frac{y^2}{1} = 1$; 6) $-(x-2)^2 + (y+3)^2 = 1$.
2	Reduce the equations of the second order curves to the canonical form and plot graphs of these curves: 1) $y^2 - 10x + 6y + 39 = 0$; 2) $4x^2 + 9y^2 - 16x + 72y + 124 = 0$; 3) $-4x^2 + y^2 - 8x - 4y - 4 = 0$; 4) $-4x^2 + y^2 + 8x - 2y - 3 = 0$.

Answers: **1.** 1) circle with center $(-4,3)$ and $R = \sqrt{5}$; 2) ellipse with center $(-2,1)$, $a = 2, b = 4$; 3) parabola with axis of symmetry parallel to Oy, center $(1,-3)$ and $p = 3$; 4) parabola with axis of symmetry parallel to Ox, center $(-1,2)$ and $p = -4$; 5) hyperbola with center $(1,0)$, $a = 3, b = 1$; 6) conjugate hyperbola with center $(2,-3)$, $a = 1, b = 1$; **2.** 1) parabola $(y+3)^2 = 10(x-3)$; 2) ellipse $\frac{(x-2)^2}{9} + \frac{(y+4)^2}{4} = 1$; 3) conjugate hyperbola $-(x+1)^2 + \frac{(y-2)^2}{4} = 1$; 4) two straight lines $y - 1 = \pm 2(x-1)$.

2.6. Individual Tasks to Chapter 2

Part 1. Vector Algebra

Hint: Braces {} *are used for vectors, parenthesis* () *are used for points.*

Task 1. Find decomposition of the vector $\vec{x}$ in the basis of vectors $\vec{g}_1,\ \vec{g}_2,\ \vec{g}_3$:

1.1. $\vec{g}_1=\{-1,\ -1,\ -4\}$, $\vec{g}_2=\{-1,\ -2,\ 3\}$, $\vec{g}_3=\{4,\ -2,\ 4\}$, $\vec{x}=\{-19,\ 6,\ -35\}$;

1.2. $\vec{g}_1=\{0,\ 5,\ 1\}$, $\vec{g}_2=\{1,\ 5,\ -2\}$, $\vec{g}_3=\{3,\ -3,\ 3\}$, $\vec{x}=\{1,\ 8,\ -14\}$;

1.3. $\vec{g}_1=\{3,\ -3,\ 3\}$, $\vec{g}_2=\{-1,\ 5,\ 3\}$, $\vec{g}_3=\{3,\ -2,\ 1\}$, $\vec{x}=\{-8,\ 26,\ 16\}$;

1.4. $\vec{g}_1=\{1,\ 1,\ -3\}$, $\vec{g}_2=\{1,\ 3,\ -5\}$, $\vec{g}_3=\{5,\ -2,\ 1\}$, $\vec{x}=\{7,\ 0,\ -5\}$;

1.5. $\vec{g}_1=\{-3,\ 1,\ 3\}$, $\vec{g}_2=\{-1,\ 5,\ -4\}$, $\vec{g}_3=\{4,\ -4,\ 4\}$, $\vec{x}=\{26,\ -36,\ 29\}$;

1.6. $\vec{g}_1=\{-4,\ 3,\ -3\}$, $\vec{g}_2=\{1,\ 5,\ 2\}$, $\vec{g}_3=\{4,\ -4,\ 4\}$, $\vec{x}=\{33,\ 2,\ 33\}$;

1.7. $\vec{g}_1=\{-2,\ -4,\ 1\}$, $\vec{g}_2=\{-1,\ 1,\ 3\}$, $\vec{g}_3=\{-4,\ -2,\ 4\}$, $\vec{x}=\{15,\ 3,\ -24\}$;

1.8. $\vec{g}_1=\{1,\ 3,\ -5\}$, $\vec{g}_2=\{1,\ 0,\ 3\}$, $\vec{g}_3=\{-5,\ -4,\ 3\}$, $\vec{x}=\{-19,\ -19,\ 23\}$;

1.9. $\vec{g}_1=\{4,\ 0,\ 2\}$, $\vec{g}_2=\{-1,\ 3,\ 1\}$, $\vec{g}_3=\{3,\ -4,\ 1\}$, $\vec{x}=\{-3,\ 11,\ 1\}$;

1.10. $\vec{g}_1=\{0,\ -1,\ -3\}$, $\vec{g}_2=\{1,\ 4,\ 5\}$, $\vec{g}_3=\{1,\ -3,\ 3\}$, $\vec{x}=\{5,\ 2,\ 28\}$;

1.11. $\vec{g}_1=\{2,\ 2,\ -1\}$, $\vec{g}_2=\{-1,\ 2,\ -3\}$, $\vec{g}_3=\{4,\ -1,\ 4\}$, $\vec{x}=\{17,\ -12,\ 26\}$;

1.12. $\vec{g}_1=\{4,\ -3,\ -5\}$, $\vec{g}_2=\{1,\ 3,\ -5\}$, $\vec{g}_3=\{2,\ -1,\ 4\}$, $\vec{x}=\{14,\ -4,\ -38\}$;

1.13. $\vec{g}_1=\{5,\ -1,\ 2\}$, $\vec{g}_2=\{-1,\ 2,\ -2\}$, $\vec{g}_3=\{-4,\ -4,\ 1\}$, $\vec{x}=\{4,\ -29,\ 20\}$;

1.14. $\vec{g}_1=\{0,\ 5,\ -2\}$, $\vec{g}_2=\{1,\ 5,\ 2\}$, $\vec{g}_3=\{1,\ -1,\ 3\}$, $\vec{x}=\{1,\ 1,\ 5\}$;

1.15. $\vec{g}_1=\{2,\ -2,\ 5\}$, $\vec{g}_2=\{-1,\ 4,\ 1\}$, $\vec{g}_3=\{5,\ -4,\ 2\}$, $\vec{x}=\{4,\ -20,\ 3\}$;

1.16. $\vec{g}_1=\{2,\ 4,\ 4\}$, $\vec{g}_2=\{1,\ 0,\ 0\}$, $\vec{g}_3=\{1,\ -4,\ 2\}$, $\vec{x}=\{-5,\ -20,\ -14\}$;

1.17. $\vec{g}_1=\{-4,\ 5,\ 3\}$, $\vec{g}_2=\{-1,\ -4,\ -2\}$, $\vec{g}_3=\{-2,\ -3,\ 3\}$, $\vec{x}=\{4,\ -43,\ -1\}$;

1.18. $\vec{g}_1=\{1,\ 3,\ -3\}$, $\vec{g}_2=\{-1,\ 4,\ -1\}$, $\vec{g}_3=\{-1,\ -4,\ 3\}$, $\vec{x}=\{3,\ -19,\ 7\}$;

1.19. $\vec{g}_1=\{1,\ 3,\ -3\}$, $\vec{g}_2=\{-1,\ 4,\ -1\}$, $\vec{g}_3=\{-1,\ -4,\ 3\}$, $\vec{x}=\{3,\ -19,\ 7\}$;

1.20. $\vec{g}_1=\{-5,\ -5,\ 1\}$, $\vec{g}_2=\{1,\ 2,\ -3\}$, $\vec{g}_3=\{-5,\ -4,\ 2\}$, $\vec{x}=\{-10,\ -11,\ 1\}$;

1.21. $\vec{g}_1=\{0,\ 2,\ 3\}$, $\vec{g}_2=\{-1,\ 4,\ -4\}$, $\vec{g}_3=\{-4,\ -2,\ 3\}$, $\vec{x}=\{-1,\ -4,\ 30\}$;

1.22. $\vec{g}_1=\{3,\ -4,\ -2\}$, $\vec{g}_2=\{1,\ -4,\ -2\}$, $\vec{g}_3=\{-5,\ -4,\ 4\}$, $\vec{x}=\{22,\ 16,\ -16\}$;

1.23. $\vec{g}_1 = \{3, -1, -3\}$, $\vec{g}_2 = \{-1, -2, 4\}$, $\vec{g}_3 = \{-5, -3, 2\}$, $\vec{x} = \{-3, 8, 21\}$;

1.24. $\vec{g}_1 = \{-4, -4, -3\}$, $\vec{g}_2 = \{1, 3, 0\}$, $\vec{g}_3 = \{-2, -3, 2\}$, $\vec{x} = \{-19, -14, -13\}$;

1.25. $\vec{g}_1 = \{-3, 2, -2\}$, $\vec{g}_2 = \{-1, 2, 3\}$, $\vec{g}_3 = \{2, -2, 2\}$, $\vec{x} = \{5, -2, 12\}$;

1.26. $\vec{g}_1 = \{-5, 1, 2\}$, $\vec{g}_2 = \{1, 2, 4\}$, $\vec{g}_3 = \{-5, -4, 1\}$, $\vec{x} = \{-9, -1, 7\}$;

1.27. $\vec{g}_1 = \{-1, -1, -2\}$, $\vec{g}_2 = \{-1, -3, 3\}$, $\vec{g}_3 = \{-5, -2, 4\}$, $\vec{x} = \{-21, -2, 18\}$;

1.28. $\vec{g}_1 = \{5, 1, -2\}$, $\vec{g}_2 = \{1, 1, -4\}$, $\vec{g}_3 = \{0, -4, 1\}$, $\vec{x} = \{-15, -11, -9\}$;

1.29. $\vec{g}_1 = \{-4, -4, 4\}$, $\vec{g}_2 = \{-1, -3, -4\}$, $\vec{g}_3 = \{1, -3, 1\}$, $\vec{x} = \{-11, -29, -10\}$;

1.30. $\vec{g}_1 = \{1, -5, 3\}$, $\vec{g}_2 = \{1, 3, 2\}$, $\vec{g}_3 = \{0, -4, 3\}$, $\vec{x} = \{-1, 9, -1\}$.

Task 2. Check the collinearity of vectors $\vec{c}_1 = a\,\vec{p} + b\vec{q}$ and $\vec{c}_2 = c\,\vec{p} + d\,\vec{q}$:

2.1. $a = 4,\ b = 4,\ c = -4,\ d = -1,\ \vec{p} = \{-1,-4,-2\},\ \vec{q} = \{-1,3,4\}$;

2.2. $a = 3,\ b = 12,\ c = 1,\ d = 4,\ \vec{p} = \{-1,0,5\},\ \vec{q} = \{1,1,5\}$;

2.3. $a = -2,\ b = 1,\ c = -5,\ d = -5,\ \vec{p} = \{1,5,-2\},\ \vec{q} = \{-1,3,-3\}$;

2.4. $a = 3,\ b = 1,\ c = 3,\ d = 3,\ \vec{p} = \{5,-5,2\},\ \vec{q} = \{1,0,1\}$;

2.5. $a = 1,\ b = -4,\ c = -3,\ d = 3,\ \vec{p} = \{-5,5,-5\},\ \vec{q} = \{-1,-4,-1\}$;

2.6. $a = -3,\ b = -4,\ c = -3,\ d = 1,\ \vec{p} = \{3,5,-4\},\ \vec{q} = \{1,4,-1\}$;

2.7. $a = 3,\ b = 6,\ c = 1,\ d = 2,\ \vec{p} = \{-4,3,-3\},\ \vec{q} = \{-1,5,2\}$;

2.8. $a = 4,\ b = 3,\ c = -3,\ d = -1,\ \vec{p} = \{-5,-2,-2\},\ \vec{q} = \{1,-4,1\}$;

2.9. $a = 1,\ b = 2,\ c = -4,\ d = 0,\ \vec{p} = \{4,-1,2\},\ \vec{q} = \{-1,4,1\}$;

2.10. $a = 3,\ b = -3,\ c = -2,\ d = 2,\ \vec{p} = \{-5,1,-4\},\ \vec{q} = \{1,1,1\}$;

2.11. $a = -2,\ b = 2,\ c = 1,\ d = 2,\ \vec{p} = \{3,1,3\},\ \vec{q} = \{-1,5,-1\}$;

2.12. $a = -3,\ b = 2,\ c = 3,\ d = -2,\ \vec{p} = \{-1,-3,4\},\ \vec{q} = \{1,5,1\}$;

2.13. $a = 1,\ b = -1,\ c = -1,\ d = -3,\ \vec{p} = \{4,2,2\},\ \vec{q} = \{-1,-1,2\}$;

2.14. $a = -3,\ b = 4,\ c = -2,\ d = -3,\ \vec{p} = \{4,2,-2\},\ \vec{q} = \{1,4,-3\}$;

2.15. $a = -5,\ b = 3,\ c = -5,\ d = 2,\ \vec{p} = \{-4,0,3\},\ \vec{q} = \{-1,-5,4\}$;

2.16. $a = 5,\ b = -1,\ c = 2,\ d = 2,\ \vec{p} = \{-2,-4,1\},\ \vec{q} = \{1,3,-2\}$;

2.17. $a = 2,\ b = -4,\ c = 0,\ d = 5,\ \vec{p} = \{-2,5,2\},\ \vec{q} = \{-1,1,2\}$;

2.18. $a=-4,\ b=-4,\ c=-5,\ d=-1,\ \vec{p}=\{2,-2,5\},\ \vec{q}=\{1,4,1\}$;

2.19. $a=5,\ b=4,\ c=3,\ d=-4,\ \vec{p}=\{2,1,2\},\ \vec{q}=\{-1,4,4\}$;

2.20. $a=0.5,\ b=1,\ c=1,\ d=2,\ \vec{p}=\{4,-4,2\},\ \vec{q}=\{1,5,-4\}$;

2.21. $a=5,\ b=-2,\ c=-4,\ d=-2,\ \vec{p}=\{-2,-4,0\},\ \vec{q}=\{-1,-4,4\}$;

2.22. $a=4,\ b=-4,\ c=-5,\ d=-2,\ \vec{p}=\{-5,1,4\},\ \vec{q}=\{1,5,0\}$;

2.23. $a=-1,\ b=-4,\ c=0.25,\ d=1,\ \vec{p}=\{3,-3,4\},\ \vec{q}=\{-1,-1,-1\}$;

2.24. $a=1,\ b=2,\ c=3,\ d=1,\ \vec{p}=\{-1,-5,-5\},\ \vec{q}=\{1,1,2\}$;

2.25. $a=-3,\ b=3,\ c=4,\ d=-2,\ \vec{p}=\{5,-3,1\},\ \vec{q}=\{-1,0,2\}$;

2.26. $a=3,\ b=3,\ c=-4,\ d=-4,\ \vec{p}=\{-2,2,1\},\ \vec{q}=\{1,-1,-4\}$;

2.27. $a=3,\ b=0,\ c=-2,\ d=-4,\ \vec{p}=\{-2,-5,-2\},\ \vec{q}=\{-1,5,-5\}$;

2.28. $a=3,\ b=2,\ c=3,\ d=-1,\ \vec{p}=\{-3,-2,4\},\ \vec{q}=\{1,-5,-3\}$;

2.29. $a=5,\ b=-1,\ c=3,\ d=1,\ \vec{p}=\{-4,-4,-3\},\ \vec{q}=\{-1,3,0\}$;

2.30. $a=-2,\ b=1,\ c=-2,\ d=-1,\ \vec{p}=\{2,2,-3\},\ \vec{q}=\{1,2,-2\}$.

Task 3. Find cosine of the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

3.1. $A(5,1,-5),\ B(-1,4,-2),\ C(-1,1,3)$;

3.2. $A(-1,-2,5),\ B(1,4,-4),\ C(1,2,-5)$;

3.3. $A(1,0,1),\ B(-1,-4,-4),\ C(-1,0,2)$;

3.4. $A(-2,0,-2),\ B(1,1,2),\ C(1,4,-1)$;

3.5. $A(-5,4,-5),\ B(-1,-3,-4),\ C(-1,3,4)$;

3.6. $A(-1,-4,1),\ B(1,-5,0),\ C(1,1,-3)$;

3.7. $A(-1,5,5),\ B(-1,-3,1),\ C(-1,5,-5)$;

3.8. $A(4,-4,-5),\ B(1,-2,2),\ C(1,-5,-2)$;

3.9. $A(-3,3,3),\ B(-1,4,5),\ C(-1,2,-3)$;

3.10. $A(5,-2,-5),\ B(1,-2,4),\ C(1,-5,-2)$;

3.11. $A(-3,-1,5),\ B(-1,5,-4),\ C(-1,-5,-5)$;

3.12. $A(2,1,-4),\ B(1,3,3),\ C(1,0,-2)$;

3.13. $A(-4,-2,5),\ B(-1,2,5),\ C(-1,1,3)$;

3.14. $A(-4,3,3)$, $B(1,0,5)$, $C(1,-4,3)$;

3.15. $A(2,5,-5)$, $B(-1,4,4)$, $C(-1,-1,2)$;

3.16. $A(-5,4,-3)$, $B(1,-5,1)$, $C(1,-1,-2)$;

3.17. $A(-1,-5,-1)$, $B(1,5,-3)$, $C(-1,-3,4)$;

3.18. $A(2,-2,-2)$, $B(1,-5,-2)$, $C(1,-1,5)$;

3.19. $A(3,-2,3)$, $B(-1,3,-1)$, $C(-1,-2,2)$;

3.20. $A(2,-2,-1)$, $B(1,-5,1)$, $C(1,3,4)$;

3.21. $A(0,-2,5)$, $B(-1,0,-5)$, $C(-1,4,4)$;

3.22. $A(4,1,4)$, $B(1,-1,3)$, $C(1,-4,-4)$;

3.23. $A(-3,-1,-1)$, $B(-1,5,3)$, $C(-1,-2,-3)$;

3.24. $A(-1,5,-3)$, $B(1,-1,0)$, $C(1,-4,1)$;

3.25. $A(3,-4,4)$, $B(-1,4,-1)$, $C(-1,-1,-4)$;

3.26. $A(-5,0,4)$, $B(1,-1,3)$, $C(1,1,-5)$;

3.27. $A(1,-2,3)$, $B(-1,0,5)$, $C(-1,1,4)$;

3.28. $A(-4,-1,2)$, $B(1,-4,-1)$, $C(1,-1,4)$;

3.29. $A(4,-4,3)$, $B(-1,1,-5)$, $C(-1,2,-2)$;

3.30. $A(-1,3,0)$, $B(1,-5,3)$, $C(1,-4,5)$.

Task 4. Find the projection of the vector $\vec{a}$ on the direction of the vector $\vec{b}$:

4.1. $\vec{a} = 6\vec{p} - \vec{q}$, $\vec{b} = \vec{p} + 2\vec{q}$; $|\vec{p}| = 8$, $|\vec{q}| = 1/2$, $(\vec{p}\hat{\ }\vec{q}) = \pi/3$.

4.2. $\vec{a} = 3\vec{p} + 4\vec{q}$, $\vec{b} = \vec{q} - \vec{p}$; $|\vec{p}| = 2{,}5$, $|\vec{q}| = 2$, $(\vec{p}\hat{\ }\vec{q}) = \pi/2$.

4.3. $\vec{a} = 7\vec{p} + \vec{q}$, $\vec{b} = \vec{p} - 3\vec{q}$; $|\vec{p}| = 3$, $|\vec{q}| = 1$, $(\vec{p}\hat{\ }\vec{q}) = 3\pi/4$.

4.4. $\vec{a} = \vec{p} + 3\vec{q}$, $\vec{b} = 3\vec{p} - \vec{q}$; $|\vec{p}| = 3$, $|\vec{q}| = 5$, $(\vec{p}\hat{\ }\vec{q}) = 2\pi/3$.

4.5. $\vec{a} = 3\vec{p} + \vec{q}$, $\vec{b} = \vec{p} - 3\vec{q}$; $|\vec{p}| = 7$, $|\vec{q}| = 2$, $(\vec{p}\hat{\ }\vec{q}) = \pi/4$.

4.6. $\vec{a} = 5\vec{p} - \vec{q}$, $\vec{b} = \vec{p} + \vec{q}$; $|\vec{p}| = 5$, $|\vec{q}| = 3$, $(\vec{p}\hat{\ }\vec{q}) = 5\pi/6$.

4.7. $\vec{a} = 3\vec{p} - 4\vec{q}$, $\vec{b} = \vec{p} + 3\vec{q}$; $|\vec{p}| = 2$, $|\vec{q}| = 3$, $(\vec{p}\hat{\ }\vec{q}) = \pi/4$.

4.8. $\vec{a} = 6\vec{p} - \vec{q},\ \vec{b} = 5\vec{q} + \vec{p};\ |\vec{p}| = 1/2,\ |\vec{q}| = 4,\ (\vec{p}\,\hat{}\,\vec{q}) = 5\pi/6.$

4.9. $\vec{a} = 2\vec{p} + 3\vec{q},\ \vec{b} = \vec{p} - 2\vec{q};\ |\vec{p}| = 2,\ |\vec{q}| = 1,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/3.$

4.10. $\vec{a} = 2\vec{p} - 3\vec{q},\ \vec{b} = 5\vec{p} + \vec{q};\ |\vec{p}| = 2,\ |\vec{q}| = 3,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/2.$

4.11. $\vec{a} = 3\vec{p} + 2\vec{q},\ \vec{b} = \vec{p} - \vec{q};\ |\vec{p}| = 10,\ |\vec{q}| = 1,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/2.$

4.12. $\vec{a} = 4\vec{p} - \vec{q},\ \vec{b} = \vec{p} + 2\vec{q};\ |\vec{p}| = 5,\ |\vec{q}| = 4,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/4.$

4.13. $\vec{a} = 2\vec{p} + 3\vec{q},\ \vec{b} = \vec{p} - 2\vec{q};\ |\vec{p}| = 6,\ |\vec{q}| = 7,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/3.$

4.14. $\vec{a} = 3\vec{p} - \vec{q},\ \vec{b} = \vec{p} + 2\vec{q};\ |\vec{p}| = 3,\ |\vec{q}| = 4,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/3.$

4.15. $\vec{a} = 2\vec{p} + 3\vec{q},\ \vec{b} = \vec{p} - 2\vec{q};\ |\vec{p}| = 2,\ |\vec{q}| = 3,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/4.$

4.16. $\vec{a} = 2\vec{p} - 3\vec{q},\ \vec{b} = 3\vec{p} + \vec{q};\ |\vec{p}| = 4,\ |\vec{q}| = 1,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/6.$

4.17. $\vec{a} = 5\vec{p} + \vec{q},\ \vec{b} = \vec{p} - 3\vec{q};\ |\vec{p}| = 1,\ |\vec{q}| = 2,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/3.$

4.18. $\vec{a} = 7\vec{p} - 2\vec{q},\ \vec{b} = \vec{p} + 3\vec{q};\ |\vec{p}| = 1/2,\ |\vec{q}| = 2,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/2.$

4.19. $\vec{a} = 6\vec{p} - \vec{q},\ \vec{b} = \vec{p} + \vec{q};\ |\vec{p}| = 3,\ |\vec{q}| = 4,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/4.$

4.20. $\vec{a} = 10\vec{p} + \vec{q},\ \vec{b} = 3\vec{p} - 2\vec{q};\ |\vec{p}| = 4,\ |\vec{q}| = 1,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/6.$

4.21. $\vec{a} = \vec{p} + 2\vec{q},\ \vec{b} = 3\vec{p} - \vec{q};\ |\vec{p}| = 1,\ |\vec{q}| = 2,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/6.$

4.22. $\vec{a} = 3\vec{p} + \vec{q},\ \vec{b} = \vec{p} - 2\vec{q};\ |\vec{p}| = 4,\ |\vec{q}| = 1,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/4.$

4.23. $\vec{a} = \vec{p} - 3\vec{q},\ \vec{b} = \vec{p} + 2\vec{q};\ |\vec{p}| = 1/5,\ |\vec{q}| = 1,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/2.$

4.24. $\vec{a} = 3\vec{p} - 2\vec{q},\ \vec{b} = \vec{p} + 5\vec{q};\ |\vec{p}| = 4,\ |\vec{q}| = 1/2,\ (\vec{p}\,\hat{}\,\vec{q}) = 5\pi/6.$

4.25. $\vec{a} = \vec{p} - 2\vec{q},\ \vec{b} = 2\vec{p} + \vec{q};\ |\vec{p}| = 2,\ |\vec{q}| = 3,\ (\vec{p}\,\hat{}\,\vec{q}) = 3\pi/4.$

4.26. $\vec{a} = \vec{p} + 3\vec{q},\ \vec{b} = \vec{p} - 2\vec{q};\ |\vec{p}| = 2,\ |\vec{q}| = 3,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/3.$

4.27. $\vec{a} = 2\vec{p} - \vec{q},\ \vec{b} = \vec{p} + 3\vec{q};\ |\vec{p}| = 3,\ |\vec{q}| = 2,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/2.$

4.28. $\vec{a} = 4\vec{p} + \vec{q},\ \vec{b} = \vec{p} - \vec{q};\ |\vec{p}| = 7,\ |\vec{q}| = 2,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/4.$

4.29. $\vec{a} = \vec{p} - 4\vec{q},\ \vec{b} = 3\vec{p} + \vec{q};\ |\vec{p}| = 1,\ |\vec{q}| = 2,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/6.$

4.30. $\vec{a} = \vec{p} + 4\vec{q},\ \vec{b} = 2\vec{p} - \vec{q};\ |\vec{p}| = 7,\ |\vec{q}| = 2,\ (\vec{p}\,\hat{}\,\vec{q}) = \pi/3.$

Task 5. Find an area of the triangle constructed on vectors $\vec{a}$ and $\vec{b}$ and the length of altitude dropped on the side of the vector $\vec{a}$:

5.1. $\vec{a}\{0,\ -5,\ -4\}$, $\vec{b}\{-1,\ 1,\ -3\}$;

5.2. $\vec{a}\{1,\ 4,\ 5\}$, $\vec{b}\{1,\ 4,\ 0\}$;

5.3. $\vec{a}\{-3,\ 2,\ -5\}$, $\vec{b}\{-1,\ -4,\ 2\}$;

5.4. $\vec{a}\{2,\ 5,\ 1\}$, $\vec{b}\{1,\ 2,\ 2\}$;

5.5. $\vec{a}\{4,\ 2,\ -1\}$, $\vec{b}\{-1,\ 5,\ 3\}$;

5.6. $\vec{a}\{-2,\ 0,\ -1\}$, $\vec{b}\{1,\ 4,\ -4\}$;

5.7. $\vec{a}\{4,\ -1,\ -3\}$, $\vec{b}\{1,\ -5,\ -4\}$;

5.8. $\vec{a}\{-4,\ 1,\ -1\}$, $\vec{b}\{-1,\ -1,\ 5\}$;

5.9. $\vec{a}\{4,\ 5,\ -4\}$, $\vec{b}\{1,\ -1,\ -1\}$;

5.10. $\vec{a}\{2,\ 3,\ 5\}$, $\vec{b}\{-1,\ 4,\ -1\}$;

5.11. $\vec{a}\{4,\ 5,\ -4\}$, $\vec{b}\{1,\ -2,\ 0\}$;

5.12. $\vec{a}\{2,\ -2,\ -1\}$, $\vec{b}\{-1,\ 3,\ 0\}$;

5.13. $\vec{a}\{0,\ -4,\ 4\}$, $\vec{b}\{1,\ -2,\ 2\}$;

5.14. $\vec{a}\{3,\ -3,\ -5\}$, $\vec{b}\{-1,\ 3,\ -2\}$;

5.15. $\vec{a}\{1,\ -5,\ 0\}$, $\vec{b}\{1,\ 2,\ 5\}$;

5.16. $\vec{a}\{1,\ 2,\ -4\}$, $\vec{b}\{-1,\ 0,\ -4\}$;

5.17. $\vec{a}\{-1,\ -1,\ 2\}$, $\vec{b}\{1,\ 5,\ 4\}$;

5.18. $\vec{a}\{-5,\ -2,\ 1\}$, $\vec{b}\{-1,\ 5,\ 5\}$;

5.19. $\vec{a}\{-4,\ 3,\ -5\}$, $\vec{b}\{1,\ 2,\ 2\}$;

5.20. $\vec{a}\{0,\ -3,\ 0\}$, $\vec{b}\{-1,\ 2,\ 1\}$;

5.21. $\vec{a}\{-2,\ 1,\ -3\}$, $\vec{b}\{1,\ 1,\ -3\}$;

5.22. $\vec{a}\{-5,\ 0,\ 0\}$, $\vec{b}\{-1,\ -3,\ -3\}$;

5.23. $\vec{a}\{0,\ -1,\ 5\}$, $\vec{b}\{1,\ 5,\ 2\}$;

5.24. $\vec{a}\{2,\ 4,\ -5\}$, $\vec{b}\{-1,\ 0,\ -1\}$;

5.25. $\vec{a}\{-3,\ 2,\ -4\}$, $\vec{b}\{1,\ -3,\ -5\}$;

5.26. $\vec{a}\{-1,\ -3,\ -5\}$, $\vec{b}\{-1,\ -1,\ 0\}$;

5.27. $\vec{a}\{-5,\ 0,\ -3\}$, $\vec{b}\{1,\ 1,\ 5\}$;

5.28. $\vec{a}\{3,\ 4,\ -5\}$, $\vec{b}\{-1,\ 0,\ 1\}$;

5.29. $\vec{a}\{-5,\ 2,\ 5\}$, $\vec{b}\{1,\ 3,\ 0\}$;

5.30. $\vec{a}\{2,\ 4,\ -3\}$, $\vec{b}\{-1,\ 0,\ -4\}$.

Task 6. Find an area of the parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$:

6.1. $\vec{a}=3\vec{p}+2\vec{q},\ \vec{b}=\vec{p}-\vec{q};\ |\vec{p}|=10,\ |\vec{q}|=1,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/2.$

6.2. $\vec{a}=4\vec{p}-\vec{q},\ \vec{b}=\vec{p}+2\vec{q};\ |\vec{p}|=5,\ |\vec{q}|=4,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

6.3. $\vec{a}=2\vec{p}+3\vec{q},\ \vec{b}=\vec{p}-2\vec{q};\ |\vec{p}|=6,\ |\vec{q}|=7,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/3.$

6.4. $\vec{a}=3\vec{p}-\vec{q},\ \vec{b}=\vec{p}+2\vec{q};\ |\vec{p}|=3,\ |\vec{q}|=4,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/3.$

6.5. $\vec{a}=2\vec{p}+3\vec{q},\ \vec{b}=\vec{p}-2\vec{q};\ |\vec{p}|=2,\ |\vec{q}|=3,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

6.6. $\vec{a}=2\vec{p}-3\vec{q},\ \vec{b}=3\vec{p}+\vec{q};\ |\vec{p}|=4,\ |\vec{q}|=1,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/6.$

6.7. $\vec{a}=5\vec{p}+\vec{q},\ \vec{b}=\vec{p}-3\vec{q};\ |\vec{p}|=1,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/3.$

6.8. $\vec{a}=7\vec{p}-2\vec{q},\ \vec{b}=\vec{p}+3\vec{q};\ |\vec{p}|=1/2,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/2.$

6.9. $\vec{a}=6\vec{p}-\vec{q},\ \vec{b}=\vec{p}+\vec{q};\ |\vec{p}|=3,\ |\vec{q}|=4,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

6.10. $\vec{a}=10\vec{p}+\vec{q},\ \vec{b}=3\vec{p}-2\vec{q};\ |\vec{p}|=4,\ |\vec{q}|=1,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/6.$

6.11. $\vec{a}=\vec{p}+2\vec{q},\ \vec{b}=3\vec{p}-\vec{q};\ |\vec{p}|=1,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/6.$

6.12. $\vec{a}=3\vec{p}+\vec{q},\ \vec{b}=\vec{p}-2\vec{q};\ |\vec{p}|=4,\ |\vec{q}|=1,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

6.13. $\vec{a}=\vec{p}-3\vec{q},\ \vec{b}=\vec{p}+2\vec{q};\ |\vec{p}|=1/5,\ |\vec{q}|=1,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/2.$

6.14. $\vec{a}=3\vec{p}-2\vec{q},\ \vec{b}=\vec{p}+5\vec{q};\ |\vec{p}|=4,\ |\vec{q}|=1/2,\ (\vec{p}\,\hat{}\,\vec{q})=5\pi/6.$

6.15. $\vec{a}=\vec{p}-2\vec{q},\ \vec{b}=2\vec{p}+\vec{q};\ |\vec{p}|=2,\ |\vec{q}|=3,\ (\vec{p}\,\hat{}\,\vec{q})=3\pi/4.$

6.16. $\vec{a}=\vec{p}+3\vec{q},\ \vec{b}=\vec{p}-2\vec{q};\ |\vec{p}|=2,\ |\vec{q}|=3,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/3.$

6.17. $\vec{a}=2\vec{p}-\vec{q},\ \vec{b}=\vec{p}+3\vec{q};\ |\vec{p}|=3,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/2.$

6.18. $\vec{a}=4\vec{p}+\vec{q},\ \vec{b}=\vec{p}-\vec{q};\ |\vec{p}|=7,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

6.19. $\vec{a}=\vec{p}-4\vec{q},\ \vec{b}=3\vec{p}+\vec{q};\ |\vec{p}|=1,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/6.$

6.20. $\vec{a}=\vec{p}+4\vec{q},\ \vec{b}=2\vec{p}-\vec{q};\ |\vec{p}|=7,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/3.$

6.21. $\vec{a}=6\vec{p}-\vec{q},\ \vec{b}=\vec{p}+2\vec{q};\ |\vec{p}|=8,\ |\vec{q}|=1/2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/3.$

6.22. $\vec{a}=3\vec{p}+4\vec{q},\ \vec{b}=\vec{q}-\vec{p};\ |\vec{p}|=2{,}5,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/2.$

6.23. $\vec{a}=7\vec{p}+\vec{q},\ \vec{b}=\vec{p}-3\vec{q};\ |\vec{p}|=3,\ |\vec{q}|=1,\ (\vec{p}\,\hat{}\,\vec{q})=3\pi/4.$

6.24. $\vec{a}=\vec{p}+3\vec{q},\ \vec{b}=3\vec{p}-\vec{q};\ |\vec{p}|=3,\ |\vec{q}|=5,\ (\vec{p}\,\hat{}\,\vec{q})=2\pi/3.$

6.25. $\vec{a}=3\vec{p}+\vec{q},\ \vec{b}=\vec{p}-3\vec{q};\ |\vec{p}|=7,\ |\vec{q}|=2,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

6.26. $\vec{a}=5\vec{p}-\vec{q},\ \vec{b}=\vec{p}+\vec{q};\ |\vec{p}|=5,\ |\vec{q}|=3,\ (\vec{p}\,\hat{}\,\vec{q})=5\pi/6.$

6.27. $\vec{a}=3\vec{p}-4\vec{q},\ \vec{b}=\vec{p}+3\vec{q};\ |\vec{p}|=2,\ |\vec{q}|=3,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/4.$

6.28. $\vec{a}=6\vec{p}-\vec{q},\ \vec{b}=5\vec{q}+\vec{p};\ |\vec{p}|=1/2,\ |\vec{q}|=4,\ (\vec{p}\,\hat{}\,\vec{q})=5\pi/6.$

6.29. $\vec{a}=2\vec{p}+3\vec{q},\ \vec{b}=\vec{p}-2\vec{q};\ |\vec{p}|=2,\ |\vec{q}|=1,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/3.$

6.30. $\vec{a}=2\vec{p}-3\vec{q},\ \vec{b}=5\vec{p}+\vec{q};\ |\vec{p}|=2,\ |\vec{q}|=3,\ (\vec{p}\,\hat{}\,\vec{q})=\pi/2.$

Task 7. Find the volume of the tetrahedron $ABCD$ and its altitude dropped from D on the base ABC.

7.1. $A(4,\ -1,\ -4),\ B(3,\ -2,\ -8),\ C(3,\ -3,\ -1),\ D(8,\ -3,\ 0);$

7.2. $A(-3,\ 4,\ -1),\ B(-3,\ 9,\ 0),\ C(-2,\ 9,\ -3),\ D(0,\ 1,\ 2);$

7.3. $A(1,\ 5,\ -2),\ B(4,\ 2,\ 1),\ C(0,\ 10,\ 1),\ D(4,\ 3,\ -1);$

7.4. $A(2,\ 0,\ 1),\ B(3,\ 1,\ -2),\ C(3,\ 3,\ -4),\ D(7,\ -2,\ 2);$

7.5. $A(-1,\ -3,\ 5),\ B(-4,\ -2,\ 8),\ C(-2,\ 2,\ 1),\ D(3,\ -7,\ 9);$

7.6. $A(-5,\ 5,\ 2),\ B(-9,\ 8,\ -1),\ C(-4,\ 10,\ 4),\ D(-1,\ 1,\ 6);$

7.7. $A(-1,\ -5,\ -2),\ B(-3,\ -9,\ -1),\ C(-2,\ -4,\ 1),\ D(-5,\ -7,\ 2);$

7.8. $A(-1,\ 2,\ 4)$, $B(0,\ 5,\ -1)$, $C(0,\ 2,\ 7)$, $D(-6,\ -2,\ 7)$;

7.9. $A(1,\ 1,\ -2)$, $B(5,\ 1,\ 0)$, $C(0,\ 4,\ -1)$, $D(4,\ -3,\ -1)$;

7.10. $A(-3,\ 2,\ 3)$, $B(-3,\ 1,\ 0)$, $C(-2,\ 6,\ 8)$, $D(-2,\ -1,\ 6)$;

7.11. $A(-1,\ -3,\ 4)$, $B(1,\ -1,\ 3)$, $C(-2,\ -1,\ 1)$, $D(3,\ -4,\ 8)$;

7.12. $A(4,\ 2,\ -2)$, $B(8,\ -1,\ -7)$, $C(5,\ 5,\ -7)$, $D(6,\ 1,\ 2)$;

7.13. $A(3,\ -5,\ 4)$, $B(8,\ -6,\ 6)$, $C(2,\ -3,\ 2)$, $D(-1,\ -9,\ 5)$;

7.14. $A(-2,\ 2,\ -1)$, $B(-2,\ 7,\ -3)$, $C(-1,\ 7,\ 1)$, $D(-1,\ 1,\ 2)$;

7.15. $A(2,\ -5,\ -1)$, $B(4,\ -7,\ 4)$, $C(1,\ -1,\ 0)$, $D(7,\ -9,\ 1)$;

7.16. $A(-4,\ 2,\ 1)$, $B(-2,\ 6,\ 5)$, $C(-3,\ 2,\ 1)$, $D(-3,\ -2,\ 3)$;

7.17. $A(-4,\ 2,\ 5)$, $B(-8,\ 7,\ 8)$, $C(-5,\ -2,\ 3)$, $D(-6,\ -1,\ 8)$;

7.18. $A(-4,\ 4,\ 4)$, $B(-5,\ -1,\ 2)$, $C(-3,\ -1,\ 5)$, $D(0,\ 0,\ 6)$;

7.19. $A(-1,\ -4,\ 0)$, $B(0,\ -1,\ -3)$, $C(-2,\ 0,\ -1)$, $D(-2,\ -8,\ 3)$;

7.20. $A(3,\ 0,\ -1)$, $B(-2,\ -5,\ 0)$, $C(4,\ 2,\ -4)$, $D(-2,\ -4,\ 1)$;

7.21. $A(5,\ -3,\ 1)$, $B(5,\ -1,\ 4)$, $C(4,\ 1,\ -3)$, $D(1,\ -5,\ 4)$;

7.22. $A(1,\ -1,\ -4)$, $B(4,\ -5,\ -6)$, $C(2,\ -5,\ -6)$, $D(-4,\ -5,\ 0)$;

7.23. $A(-5,\ 3,\ -3)$, $B(-2,\ 2,\ -6)$, $C(-6,\ 1,\ 1)$, $D(-10,\ 0,\ -1)$;

7.24. $A(5,\ 3,\ 1)$, $B(1,\ -1,\ -2)$, $C(6,\ 6,\ 1)$, $D(3,\ 0,\ 3)$;

7.25. $A(-1,\ 2,\ 2)$, $B(-4,\ 4,\ 0)$, $C(-2,\ 4,\ 5)$, $D(1,\ 0,\ 4)$;

7.26. $A(1,\ 1,\ 1)$, $B(-4,\ 2,\ 3)$, $C(2,\ 3,\ 5)$, $D(-4,\ -3,\ 2)$;

7.27. $A(-2,\ -2,\ 5)$, $B(-3,\ -3,\ 3)$, $C(-3,\ -5,\ 8)$, $D(-7,\ -4,\ 9)$;

7.28. $A(-4,\ 5,\ 3)$, $B(1,\ 6,\ 1)$, $C(-3,\ 6,\ -1)$, $D(-4,\ 1,\ 4)$;

7.29. $A(2,\ 5,\ 2)$, $B(-2,\ 1,\ 6)$, $C(1,\ 2,\ -2)$, $D(3,\ 2,\ 3)$;

7.30. $A(-2,\ 1,\ 1)$, $B(-1,\ -4,\ 4)$, $C(-1,\ 4,\ 3)$, $D(-2,\ -3,\ 4)$.

Part 2. Surfaces and Lines

Task 8. Find an equation of the plane passing through the points M_1, M_2, M_3 and the distance from the point M_0 to this plane.

8.1. $M_1(-1,\ -4,\ 0)$, $M_2(0,\ -1,\ -3)$, $M_3(-2,\ 0,\ -1)$, $M_4(-2,\ -8,\ 3)$;

8.2. $M_1(3,\ 0,\ -1)$, $M_2(-2,\ -5,\ 0)$, $M_3(4,\ 2,\ -4)$, $M_4(-2,\ -4,\ 1)$;

8.3. $M_1(5,\ -3,\ 1)$, $M_2(5,\ -1,\ 4)$, $M_3(4,\ 1,\ -3)$, $M_4(1,\ -5,\ 4)$;

8.4. $M_1(1,\ -1,\ -4)$, $M_2(4,\ -5,\ -6)$, $M_3(2,\ -5,\ -6)$, $M_4(-4,\ -5,\ 0)$;

8.5. $M_1(-5,\ 3,\ -3)$, $M_2(-2,\ 2,\ -6)$, $M_3(-6,\ 1,\ 1)$, $M_4(-10,\ 0,\ -1)$;

8.6. $M_1(5,\ 3,\ 1)$, $M_2(1,\ -1,\ -2)$, $M_3(6,\ 6,\ 1)$, $M_4(3,\ 0,\ 3)$;

8.7. $M_1(-1,\ 2,\ 2)$, $M_2(-4,\ 4,\ 0)$, $M_3(-2,\ 4,\ 5)$, $M_4(1,\ 0,\ 4)$;

8.8. $M_1(1,\ 1,\ 1)$, $M_2(-4,\ 2,\ 3)$, $M_3(2,\ 3,\ 5)$, $M_4(-4,\ -3,\ 2)$;

8.9. $M_1(-2,\ -2,\ 5)$, $M_2(-3,\ -3,\ 3)$, $M_3(-3,\ -5,\ 8)$, $M_4(-7,\ -4,\ 9)$;

8.10. $M_1(-4,\ 5,\ 3)$, $M_2(1,\ 6,\ 1)$, $M_3(-3,\ 6,\ -1)$, $M_4(-4,\ 1,\ 4)$;

8.11. $M_1(2,\ 5,\ 2)$, $M_2(-2,\ 1,\ 6)$, $M_3(1,\ 2,\ -2)$, $M_4(3,\ 2,\ 3)$;

8.12. $M_1(-2,\ 1,\ 1)$, $M_2(-1,\ -4,\ 4)$, $M_3(-1,\ 4,\ 3)$, $M_4(-2,\ -3,\ 4)$;

8.13. $M_1(4,\ -1,\ -4)$, $M_2(3,\ -2,\ -8)$, $M_3(3,\ -3,\ -1)$, $M_4(8,\ -3,\ 0)$;

8.14. $M_1(-3,\ 4,\ -1)$, $M_2(-3,\ 9,\ 0)$, $M_3(-2,\ 9,\ -3)$, $M_4(0,\ 1,\ 2)$;

8.15. $M_1(-4,\ 4,\ 4)$, $M_2(-5,\ -1,\ 2)$, $M_3(-3,\ -1,\ 5)$, $M_4(0,\ 0,\ 6)$;

8.16. $M_1(2,\ -5,\ -1)$, $M_2(4,\ -7,\ 4)$, $M_3(1,\ -1,\ 0)$, $M_4(7,\ -9,\ 1)$;

8.17. $M_1(1,\ 5,\ -2)$, $M_2(4,\ 2,\ 1)$, $M_3(0,\ 10,\ 1)$, $M_4(4,\ 3,\ -1)$;

8.18. $M_1(2,\ 0,\ 1)$, $M_2(3,\ 1,\ -2)$, $M_3(3,\ 3,\ -4)$, $M_4(7,\ -2,\ 2)$;

8.19. $M_1(-4,\ 2,\ 1)$, $M_2(-2,\ 6,\ 5)$, $M_3(-3,\ 2,\ 1)$, $M_4(-3,\ -2,\ 3)$;

8.20. $M_1(-1,\ -3,\ 5)$, $M_2(-4,\ -2,\ 8)$, $M_3(-2,\ 2,\ 1)$, $M_4(3,\ -7,\ 9)$;

8.21. $M_1(-5,\ 5,\ 2)$, $M_2(-9,\ 8,\ -1)$, $M_3(-4,\ 10,\ 4)$, $M_4(-1,\ 1,\ 6)$;

8.22. $M_1(-4,\ 2,\ 5)$, $M_2(-8,\ 7,\ 8)$, $M_3(-5,\ -2,\ 3)$, $M_4(-6,\ -1,\ 8)$;

8.23. $M_1(-4,\ 4,\ 4)$, $M_2(-5,\ -1,\ 2)$, $M_3(-3,\ -1,\ 5)$, $M_4(0,\ 0,\ 6)$;

8.24. $M_1(-1,\ 2,\ 4)$, $M_2(0,\ 5,\ -1)$, $M_3(0,\ 2,\ 7)$, $M_4(-6,\ -2,\ 7)$;

8.25. $M_1(1,\ 1,\ -2)$, $M_2(5,\ 1,\ 0)$, $M_3(0,\ 4,\ -1)$, $M_4(4,\ -3,\ -1)$;

8.26. $M_1(-3,\ 2,\ 3)$, $M_2(-3,\ 1,\ 0)$, $M_3(-2,\ 6,\ 8)$, $M_4(-2,\ -1,\ 6)$;

8.27. $M_1(-1,\ -3,\ 4)$, $M_2(1,\ -1,\ 3)$, $M_3(-2,\ -1,\ 1)$, $M_4(3,\ -4,\ 8)$;

8.28. $M_1(4,\ 2,\ -2)$, $M_2(8,\ -1,\ -7)$, $M_3(5,\ 5,\ -7)$, $M_4(6,\ 1,\ 2)$;

8.29. $M_1(3,\ -5,\ 4)$, $M_2(8,\ -6,\ 6)$, $M_3(2,\ -3,\ 2)$, $M_4(-1,\ -9,\ 5)$;

8.30. $M_1(-2,\ 2,\ -1)$, $M_2(-2,\ 7,\ -3)$, $M_3(-1,\ 7,\ 1)$, $M_4(-1,\ 1,\ 2)$.

Task 9. Find the canonical equations of the given straight line:

9.1. $\begin{cases} 3x+y+z-2=0 \\ 2x-y-3z+6=0 \end{cases}$

9.2. $\begin{cases} x-3y+2z+3=0 \\ x+3y+z+14=0 \end{cases}$

9.3. $\begin{cases} x-2y-3z-4=0 \\ 2x+2y-z-8=0 \end{cases}$

9.4. $\begin{cases} 2x+y+z-2=0 \\ 2x-y-2z+2=0 \end{cases}$

9.5. $\begin{cases} x+3y+z+6=0 \\ x-3y-2z+3=0 \end{cases}$

9.6. $\begin{cases} x+y-z-6=0 \\ 2x-y+2z=0 \end{cases}$

9.7. $\begin{cases} x+5y+2z+11=0 \\ x-2y-z-1=0 \end{cases}$

9.8. $\begin{cases} x+4y-2z+1=0 \\ 2x-4y+3z+4=0 \end{cases}$

9.9. $\begin{cases} 5x+y-3z+4=0 \\ 2x-y+2z+2=0 \end{cases}$

9.10. $\begin{cases} 2x-y-z-2=0 \\ 3x-2y+z+4=0 \end{cases}$

9.11. $\begin{cases} x+y-3z+2=0 \\ 2x-y+z-8=0 \end{cases}$

9.12. $\begin{cases} x+3y-2z-1=0 \\ 2x-3y+z+6=0 \end{cases}$

9.13. $\begin{cases} 2x-5y-4z-2=0 \\ x+7y-z-5=0 \end{cases}$

9.14. $\begin{cases} 2x-y-3z-1=0 \\ x+y+z+10=0 \end{cases}$

9.15. $\begin{cases} x-5y-4z+8=0 \\ x+5y+3z+4=0 \end{cases}$

9.16. $\begin{cases} 2x+y-z-5=0 \\ 2x-5y+2z+5=0 \end{cases}$

9.17. $\begin{cases} 2x-3y+z+6=0 \\ 2x-3y-2z+3=0 \end{cases}$

9.18. $\begin{cases} 5x+y+2z+4=0 \\ x-2y-3z+2=0 \end{cases}$

9.19. $\begin{cases} x+y+z+2=0 \\ 2x-y-3z-8=0 \end{cases}$

9.20. $\begin{cases} x+y-3z-2=0 \\ x-y+z+6=0 \end{cases}$

9.21. $\begin{cases} x+3y-2z-2=0 \\ x-y+z+2=0 \end{cases}$

9.22. $\begin{cases} 2x+5y-z+11=0 \\ x-y+2z-1=0 \end{cases}$

9.23. $\begin{cases} 2x-y+z-2=0 \\ x-2y-z+4=0 \end{cases}$

9.24. $\begin{cases} 5x-y-z-2=0 \\ x+7y-4z-5=0 \end{cases}$

9.25. $\begin{cases} x+5y+2z-5=0 \\ x-5y-z+5=0 \end{cases}$

9.26. $\begin{cases} 4x-3y+z+2=0 \\ x+3y+2z+14=0 \end{cases}$

9.27. $\begin{cases} 2x+5y-2z+6=0 \\ x-3y+z+3=0 \end{cases}$

9.28. $\begin{cases} x+4y+3z+1=0 \\ 3x-4y-2z+4=0 \end{cases}$

9.29. $\begin{cases} x+3y+z-1=0 \\ 2x-3y-2z+6=0 \end{cases}$

9.30. $\begin{cases} 2x-5y+3z+8=0 \\ 2x+5y-4z+4=0 \end{cases}$

Task 10. Find a point of intersection of the given straight line and plane:

10.1. $\dfrac{x-3}{1}=\dfrac{y-1}{-1}=\dfrac{z+5}{0}, \quad x+7y+3z+11=0;$

10.2. $\dfrac{x-5}{-1}=\dfrac{y+3}{5}=\dfrac{z-1}{2}, \quad 3x+7y-5z-11=0;$

10.3. $\dfrac{x-1}{7}=\dfrac{y-2}{1}=\dfrac{z-6}{-1}, \quad 4x+y-6z-5=0;$

10.4. $\dfrac{x-3}{1}=\dfrac{y+2}{-1}=\dfrac{z-8}{0}, \quad 5x+9y+4z-25=0;$

10.5. $\dfrac{x+1}{-2}=\dfrac{y}{0}=\dfrac{z+1}{3}, \quad x+4y+13z-23=0;$

10.6. $\dfrac{x-1}{6}=\dfrac{y-3}{1}=\dfrac{z+5}{3}, \quad 3x-2y+5z-3=0;$

10.7. $\dfrac{x-2}{4}=\dfrac{y-1}{-3}=\dfrac{z+3}{-2}, \quad 3x-y+4z=0;$

10.8. $\dfrac{x-1}{2}=\dfrac{y+2}{-5}=\dfrac{z-3}{-2}, \quad x+2y-5z+16=0;$

10.9. $\frac{x-1}{1}=\frac{y-3}{0}=\frac{z+2}{-2}$, $3x-7y-2z+7=0$;

10.10. $\frac{x+3}{0}=\frac{y-2}{-3}=\frac{z+5}{11}$, $5x+7y+9z-32=0$;

10.11. $\frac{x-1}{2}=\frac{y-1}{-1}=\frac{z+2}{3}$, $4x+2y-z-11=0$;

10.12. $\frac{x-1}{1}=\frac{y+1}{0}=\frac{z-1}{-1}$, $3x-2y-4z-8=0$;

10.13. $\frac{x+2}{-1}=\frac{y-1}{1}=\frac{z+3}{2}$, $x+2y-z-2=0$;

10.14. $\frac{x+3}{1}=\frac{y-2}{-5}=\frac{z+2}{3}$, $5x-y+4z+3=0$;

10.15. $\frac{x-2}{2}=\frac{y-2}{-1}=\frac{z-4}{3}$, $x+3y+5z-42=0$;

10.16. $\frac{x-3}{-1}=\frac{y-4}{5}=\frac{z-4}{2}$, $7x+y+4z-47=0$;

10.17. $\frac{x+3}{2}=\frac{y-1}{3}=\frac{z-1}{5}$, $2x+3y+7z-52=0$;

10.18. $\frac{x-3}{2}=\frac{y+1}{3}=\frac{z+3}{2}$, $3x+4y+7z-16=0$;

10.19. $\frac{x-5}{-2}=\frac{y-2}{0}=\frac{z+4}{-1}$, $2x-5y+4z+24=0$;

10.20. $\frac{x-1}{8}=\frac{y-8}{-5}=\frac{z+5}{12}$, $x-2y-3z+18=0$;

10.21. $\frac{x-2}{-1}=\frac{y-3}{-1}=\frac{z+1}{4}$, $x+2y+3z-14=0$;

10.22. $\frac{x+1}{3}=\frac{y-3}{-4}=\frac{z+1}{5}$, $x+2y-5z+20=0$;

10.23. $\frac{x-1}{-1}=\frac{y+5}{4}=\frac{z-1}{2}$, $x-3y+7z-24=0$;

10.24. $\frac{x-1}{1}=\frac{y}{0}=\frac{z+3}{2}$, $2x-y+4z=0$;

10.25. $\frac{x-5}{1}=\frac{y-3}{-1}=\frac{z-2}{0}$, $3x+y-5z-12=0$;

10.26. $\frac{x+1}{-3}=\frac{y+2}{2}=\frac{z-3}{-2}$, $x+3y-5z+9=0$;

10.27. $\frac{x-1}{-2}=\frac{y-2}{1}=\frac{z+1}{-1}$, $x-2y+5z+17=0$;

10.28. $\frac{x-1}{2}=\frac{y-2}{0}=\frac{z-4}{1}$, $x-2y+4z-19=0$;

10.29. $\frac{x+2}{-1}=\frac{y-1}{1}=\frac{z+4}{-1}$, $2x-y+3z+23=0$;

10.30. $\frac{x+2}{1}=\frac{y-2}{0}=\frac{z+3}{0}$, $2x-3y-5z-7=0$.

Task 11. Find a projection of point M on the given plane (for odd variants) or straight line (for even variants):

11.1. $M(0,\ -3,\ -2)$, $3x+y-z-1=0$;

11.2. $M(1,\ 0,\ -1)$, $\frac{x-2}{0}=\frac{y-1}{-2}=\frac{z-1}{1}$;

11.3. $M(1,\ 1,\ 1)$, $x+y-z=0$;

11.4. $M(3,\ -3,\ -1)$, $\frac{x-2}{1}=\frac{y+2}{1}=\frac{z-2}{-1}$;

11.5. $M(1,\ 0,\ -1)$, $2x+y+4=0$;

11.6. $M(-2,\ -3,\ 0)$, $\frac{x-2}{3}=\frac{y}{2}=\frac{z}{1}$;

11.7. $M(-2,\ -3,\ 0)$, $3x+4z-1=0$;

11.8. $M(2,\ -2,\ -3)$, $\frac{x}{3}=\frac{y+3}{1}=\frac{z-2}{1}$;

11.9. $M(0,\ 2,\ 1)$, $x+y+z=0$;

11.10. $M(-1,\ 0,\ 1)$, $\frac{x+1}{2}=\frac{y}{-2}=\frac{z-2}{-1}$;

11.11. $M(3,\ 3,\ 3)$, $x-y-z-1=0$;

11.12. $M(3,\quad 3,\quad 3),\quad \dfrac{x+1}{1}=\dfrac{y+1}{-2}=\dfrac{z-1}{5};$

11.13. $M(2,\quad -2,\quad -3),\quad x-y-z+2=0;$

11.14. $M(-1,\quad 2,\quad 0),\quad \dfrac{x+0,5}{1}=\dfrac{y+0,7}{-0,2}=\dfrac{z-2}{2};$

11.15. $M(0,\quad -3,\quad -2),\quad -x+y+5z=0;$

11.16. $M(2,\quad -2,\quad -3),\quad \dfrac{x-1}{-1}=\dfrac{y+0,5}{0}=\dfrac{z+1,5}{0};$

11.17. $M(-1,\quad 0,\quad -1),\quad 2x+6y-2z+11=0;$

11.18. $M(0,\quad -3,\quad -2),\quad \dfrac{x-0,5}{0}=\dfrac{y+1,5}{-1}=\dfrac{z-1,5}{1};$

11.19. $M(2,\quad 1,\quad 0),\quad y+z+2=0;$

11.20. $M(2,\quad 1,\quad 0),\quad \dfrac{x-3}{0}=\dfrac{y-1}{1}=\dfrac{z-2}{1};$

11.21. $M(2,\quad -1,\quad 1),\quad x-y+2z-2=0;$

11.22. $M(-1,\quad 2,\quad 0),\quad \dfrac{x-1}{1}=\dfrac{y-4}{1}=\dfrac{z-2}{-1};$

11.23. $M(1,\quad 2,\quad 3),\quad 2x+10y+10z-1=0;$

11.24. $M(2,\quad -1,\quad 1),\quad \dfrac{x-4}{1}=\dfrac{y-5}{-2}=\dfrac{z-2}{1};$

11.25. $M(1,\quad 0,\quad -1),\quad 2y+4z-1=0;$

11.26. $M(1,\quad 1,\quad 1),\quad \dfrac{x+1}{3}=\dfrac{y+1}{2}=\dfrac{z-2}{1};$

11.27. $M(-2,\quad -3,\quad 0),\quad x+5y+4=0;$

11.28. $M(1,\quad 2,\quad 3),\quad \dfrac{x-1}{2}=\dfrac{y-3}{-5}=\dfrac{z}{5};$

11.29. $M(-1,\quad 0,\quad 1),\quad 2x+4y-3=0;$

11.30. $M(0,\quad -3,\quad -2),\quad \dfrac{x}{-1}=\dfrac{y+1}{-2}=\dfrac{z-2}{2}.$

Task 12. Coordinates of the triangle vertices are given. Find:

1) the equations of the triangle sides;
2) the equation of the triangle median dropped from the vertex B;
3) the equation of the altitude dropped from the vertex A;
4) the equation of the bisector of angle $\angle C$;
5) the equation of the middle line of the triangle parallel to the side CB;
6) the length of the altitude dropped from the vertex B;
7) the area of this trianlgle.

Hint: the equation of a side (median, altitude, etc.) means the equation of a straight line passing through this side.

12.1. $A(4,4)$, $B(4,-3)$, $C(1,1)$;
12.2. $A(1,3)$, $B(3,-1)$, $C(5,3)$;
12.3. $A(5,4)$, $B(2,-4)$, $C(1,1)$;
12.4. $A(-5,1)$, $B(1,3)$, $C(5,4)$;
12.5. $A(-1,2)$, $B(3,-5)$, $C(4,2)$;
12.6. $A(0,2)$, $B(4,-1)$, $C(-2,1)$;
12.7. $A(5,1)$, $B(2,-5)$, $C(-3,3)$;
12.8. $A(-2,2)$, $B(2,2)$, $C(2,4)$;
12.9. $A(-1,1)$, $B(3,2)$, $C(-4,3)$;
12.10. $A(2,1)$, $B(3,-1)$, $C(5,4)$;
12.11. $A(4,1)$, $B(4,2)$, $C(5,2)$;
12.12. $A(-3,3)$, $B(1,1)$, $C(-5,3)$;
12.13. $A(2,4)$, $B(4,1)$, $C(-3,2)$;
12.14. $A(5,3)$, $B(2,-5)$, $C(-2,3)$;
12.15. $A(1,4)$, $B(3,-4)$, $C(5,3)$;
12.16. $A(-1,1)$, $B(2,5)$, $C(5,2)$;
12.17. $A(3,3)$, $B(2,3)$, $C(-2,1)$;
12.18. $A(-2,1)$, $B(2,0)$, $C(0,3)$;
12.19. $A(4,1)$, $B(3,2)$, $C(-2,2)$;
12.20. $A(4,4)$, $B(2,2)$, $C(5,4)$;
12.21. $A(-3,3)$, $B(1,4)$, $C(2,1)$;
12.22. $A(-3,3)$, $B(3,-4)$, $C(4,3)$;
12.23. $A(1,4)$, $B(3,1)$, $C(-5,1)$;
12.24. $A(4,1)$, $B(3,-1)$, $C(-2,1)$;
12.25. $A(1,3)$, $B(3,4)$, $C(-1,3)$;
12.26. $A(-3,3)$, $B(4,-4)$, $C(2,1)$;
12.27. $A(5,4)$, $B(2,-4)$, $C(0,1)$;
12.28. $A(0,2)$, $B(3,-3)$, $C(4,3)$;
12.29. $A(0,4)$, $B(3,-3)$, $C(-5,1)$;
12.30. $A(1,4)$, $B(2,-4)$, $C(4,1)$.

Task 13. Reduce the equations of the second order curves to the canonical form and plot graphs of these curves.

13.1.

a) $x^2-6x-4y+29=0$;

b) $9x^2+4y^2-18x-8y-23=0$.

13.2.

a) $x^2+y^2+2x-6y+1=0$;

b) $3y^2+5x+6y+13=0$.

13.3.

a) $x^2 - 4y^2 + 6x + 16y - 11 = 0$;

b) $x^2 + 4x + 3y = 0$.

13.4.

a) $x^2 + 4y^2 + 18x + 16y - 11 = 0$;

b) $2y^2 - 4y + 5x = 0$.

13.5.

a) $y^2 - 2x + 8y + 10 = 0$;

b) $4x^2 - y^2 - 2y - 5 = 0$.

13.6.

a) $7x^2 - 2y^2 + 28x + 14 = 0$;

b) $2x^2 - 8x - y = 0$.

13.7.

a) $x^2 + 2y^2 + 4y - 6 = 0$;

b) $4y^2 - 8y + x = 0$.

13.8.

a) $5x^2 + 9y^2 - 30x + 18y + 9 = 0$;

b) $y^2 - 2x + 4y + 2 = 0$.

13.9.

a) $4x^2 - 9y^2 - 8x - 36y - 68 = 0$;

b) $y + x^2 + 2x = 0$.

13.10.

a) $x^2 - 10x - 4y - 3 = 0$;

b) $4x^2 + 16y^2 + 24x - 28 = 0$.

13.11.

a) $-x^2 + 3y^2 + 6x - 12y = 0$;

b) $4y^2 + 2x + 8y + 1 = 0$.

13.12.

a) $x^2 + y^2 - 6x - 8y + 9 = 0$;

b) $3x^2 + 12x + 16y - 12 = 0$.

13.13.

a) $2y^2 + x - 8y + 3 = 0$;

b) $x^2 - y^2 + x - y + 1 = 0$.

13.14.

a) $x^2 - 4y^2 - 2x - 8y + 13 = 0$;

b) $3y^2 - 12x - 6y + 11 = 0$.

13.15.

a) $x^2 + y^2 - 14x - 8y + 40 = 0$;

b) $10x - 5x^2 - 2y + 3 = 0$.

13.16.

a) $x^2 - y^2 - 6x + 4y + 6 = 0$;

b) $2y^2 + x - 4y - 8 = 0$.

13.17.

a) $-3x^2 + y^2 - 6x - 4y - 11 = 0$;

b) $x^2 + x + 2y - 1 = 0$.

13.18.

a) $9x^2 + 25y^2 - 18x + 100y - 116 = 0$;

b) $x^2 - 2y + 6x + 1 = 0$.

13.19.

a) $x^2 - 4y^2 - 8x + 12 = 0$;

b) $3x^2 - 2y + 6x + 1 = 0$.

13.20.

a) $5x^2 + 9y^2 - 30x + 18y + 9 = 0$;

b) $3x^2 + y + 6x = 0$.

13.21.

a) $x^2 - y^2 - 4x + 2y + 2 = 0$;

b) $3y^2 + 2x + 3y = 0$.

13.22.

a) $16x^2 + 25y^2 + 64x - 50y - 311 = 0$;

b) $3y^2 + 2x + 3y + 2 = 0$.

13.23.

a) $2x^2 - 8x + y + 5 = 0$;

b) $4x^2 + y^2 + 8x - 2y - 11 = 0$.

13.24.

a) $9x^2 + 4y^2 - 18x - 8y - 23 = 0$;

b) $x^2 + 6x - 2y + 1 = 0$.

13.25.

a) $x^2 + y^2 - 2x + 6y - 6 = 0$;

b) $2y - 5x^2 + 10x = 0$.

13.26.

a) $9x^2 - 16y^2 + 90x + 32y - 367 = 0$;

b) $x^2 + 6x + 2y = 0$.

13.27.

a) $4x^2 + 3y^2 - 8x + 12y - 32 = 0$;

b) $5y^2 + 10y + x = 0$.

13.28.

a) $x^2 + y^2 + 4x - 5 = 0$;

b) $4x^2 + 8x - y = 0$.

13.29.

a) $4x^2 - y^2 - 8x - 6y - 25 = 0$;

b) $y = 8x - 2x^2 - 5$.

13.30.

a) $4x^2 - 8x + y + 7 = 0$;

b) $x^2 - 4y^2 + 6x + 16y - 11 = 0$.

Task 14. Reduce an equation of the second order curve to the canonical form and plot graph of this curve.

14.1. $-x^2 - y^2 + 4xy + 2x - 4y + 1 = 0$.

14.2. $x^2 + y^2 - 8xy - 20x + 20y + 1 = 0$.

14.3. $2x^2 + 2y^2 - 2xy - 2x - 2y + 1 = 0$.

14.4. $3x^2 + 3y^2 - 2xy - 6x + 2y + 1 = 0$.

14.5. $4xy + 4x - 4y = 0$.

14.6. $4xy + 4x + 4y + 1 = 0$.

14.7. $-2x^2 - 2y^2 + 2xy - 6x + 6y + 3 = 0$.

14.8. $3x^2 + 3y^2 - 4xy + 6x - 4y - 7 = 0$.

14.9. $-3x^2 - 3y^2 + 4xy - 6x + 4y + 2 = 0$.

14.10. $-4xy - 4x + 4y + 6 = 0$.

14.11. $-2xy - 2x - 2y + 1 = 0$.

14.12. $5x^2+5y^2-2xy+10x-2y+1=0$.

14.13. $-x^2-y^2-4xy-4x-2y+2=0$.

14.14. $2x^2+2y^2+4xy+8x+8y+1=0$.

14.15. $-4x^2-4y^2+2xy+10x-10y+1=0$.

14.16. $-x^2-y^2+2xy+2x-2y+1=0$.

14.17. $4xy+4x-4y-2=0$.

14.18. $2x^2+2y^2-4xy-8x+8y+1=0$.

14.19. $x^2+y^2+2xy-8x-8y+1=0$.

14.20. $3x^2+3y^2+2xy-12x-4y+1=0$.

14.21. $x^2+y^2+4xy-8x-4y+1=0$.

14.22. $-4xy+8x+8y+1=0$.

14.23. $x^2+y^2-2xy-2x+2y-7=0$.

14.24. $2x^2+2y^2-2xy+6x-6y-6=0$.

14.25. $2xy+2x+2y-3=0$.

14.26. $x^2+y^2+4xy+4x+2y-5=0$.

14.27. $4x^2+4y^2+2xy+12x+12y+1=0$.

14.28. $4xy+4x-4y+4=0$.

14.29. $3x^2+3y^2+4xy+8x+12y+1=0$.

14.30. $3x^2+3y^2-4xy+4x+4y+1=0$.

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